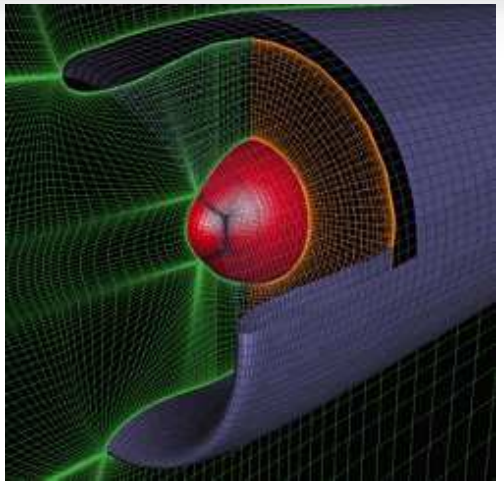
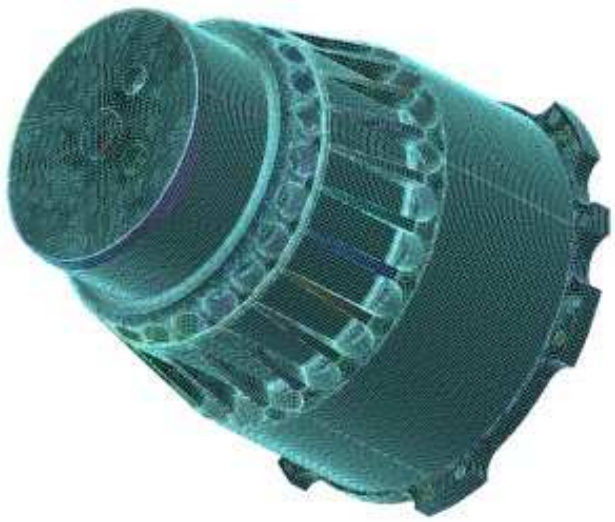
An Introduction to The Finite Element Methods

Fathi El-Yafi
Senior Product Engineer
Product Department of EXA Corporation

Finite Element Methods

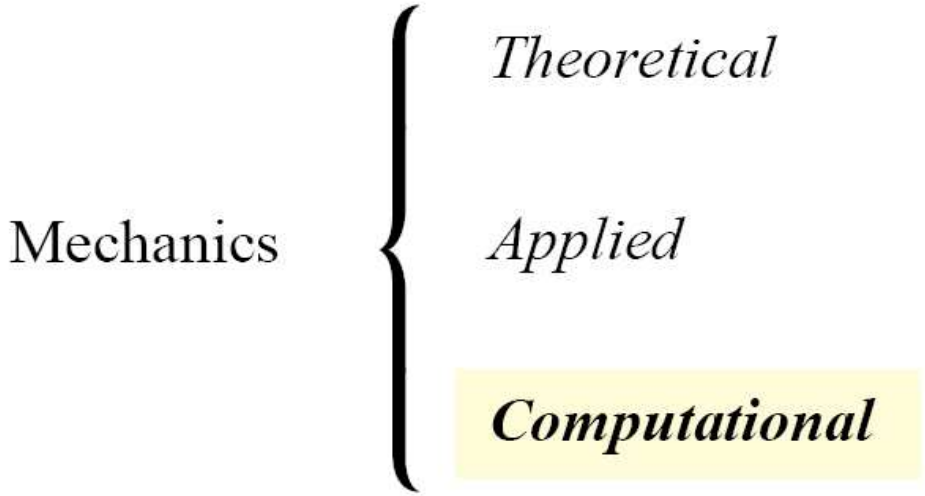
1. **Finite Element Basic Concepts**
   - Overview
   - The Direct Stiffness Method
2. **Finite Elements Modeling**
   - Introduction
   - Mesh, Loads and BCs
3. **Variational Formulation**
   - Bar Element
   - Plane Beam Element
4. **The Plane Stress Problem**
5. **Three Node Plane Stress Triangles**
6. **Isoparametric Representation**
7. **Convergence Requirements**





Where the Course Fits

The field of Mechanics can be subdivided into 3 major areas:



Computational Mechanics

Branches of ***Computational Mechanics*** can be distinguished according to the physical focus of attention

Computational Mechanics {

Nano and Micromechanics

Continuum Mechanics:
Solids and Structures
Fluids
Multiphysics

Systems

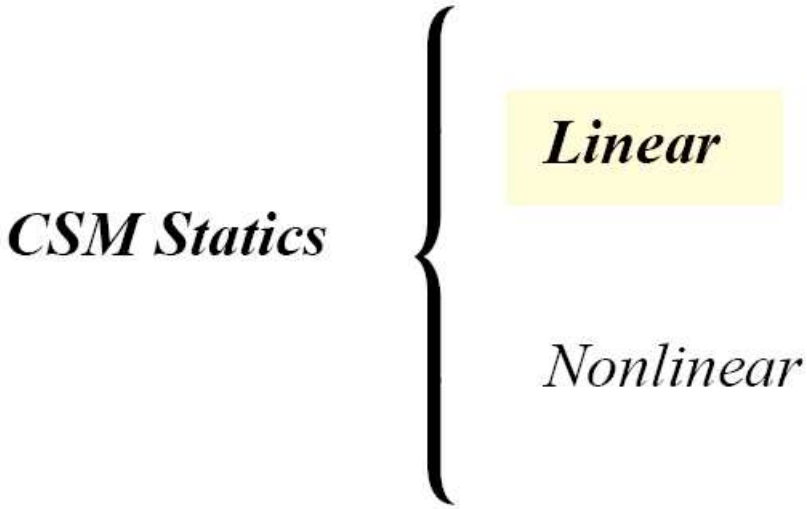
Computational Solid and Structural Mechanics

A convenient subdivision of problems in Computational Solid and Structural Mechanics (CSM) is

Computational Solid and Structural Mechanics (CSM) { ***Statics*** ; *Dynamics*

CSM Statics

A further subdivision of problems in CSM Statics is



CSM Linear Statics

For the numerical simulation on the computer we must now chose a ***spatial discretization method:***

CSM Linear Statics {

- ***Finite Element Method***
- *Finite Difference Method*
- *Boundary Element Method*
- *Finite Volume Method*
- *Spectral Method*
- *Mesh-Free Method*

CSM Linear Statics by FEM

Having selected the FEM for ***discretization,*** we must next pick a ***formulation and a solution method:***

Formulation of FEM Model {
- ***Displacement***
- *Equilibrium*
- *Mixed*
- *Hybrid*

Solution of FEM Model {
- ***Stiffness***
- *Flexibility*
- *Mixed*

Summarizing: This Course Covers

Computational structural mechanics

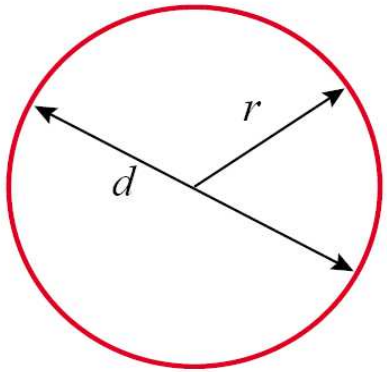
Linear static problems

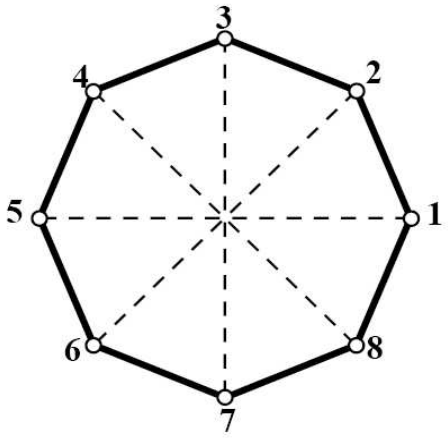
Spatially discretized by displacement-formulated FEM

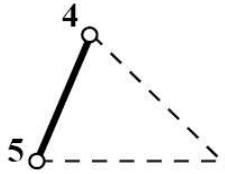
Solved by the stiffness method

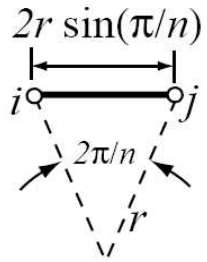
What is a Finite Element?

Archimedes' problem (*circa* 250 B.C.): rectification of the circle as limit of inscribed regular polygons







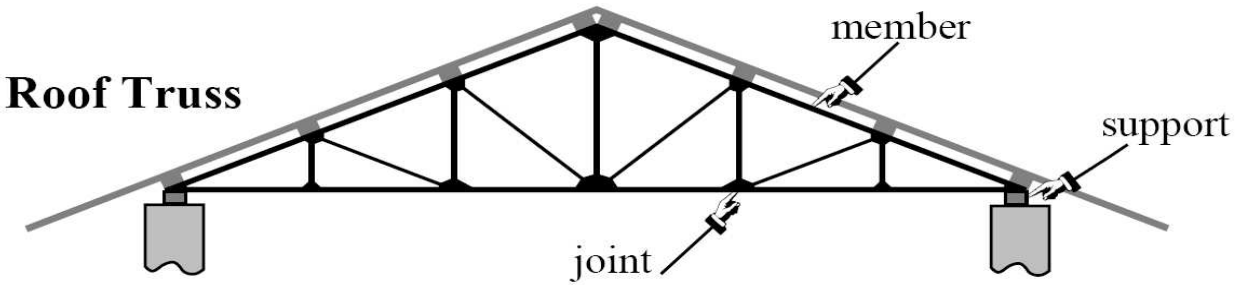


Computing π "by Archimedes FEM"

n	$\pi_n = n \sin(\pi/n)$	Extrapolated by Wynn-ε	Exact π to 16 places
1	0.000000000000000		
2	2.000000000000000		
4	2.828427124746190	3.414213562373096	
8	3.061467458920718		
16	3.121445152258052	3.141418327933211	
32	3.136548490545939		
64	3.140331156954753	3.141592658918053	
128	3.141277250932773		
256	3.141513801144301	3.141592653589786	3.141592653589793

FEM: Overview

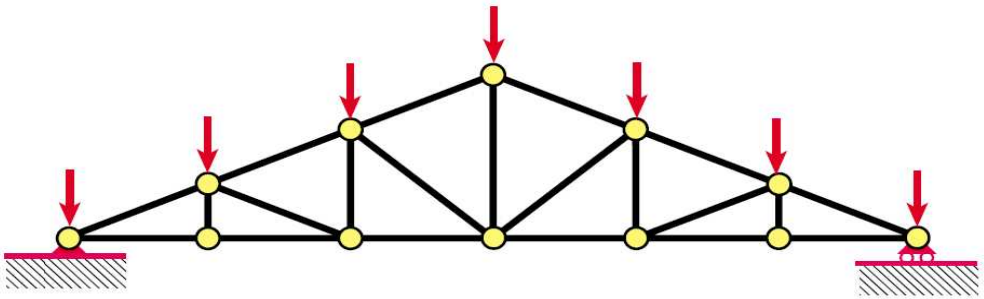
The Idealization Process for a Simple Structure



Physical Model

IDEALIZATION & DISCRETIZATION

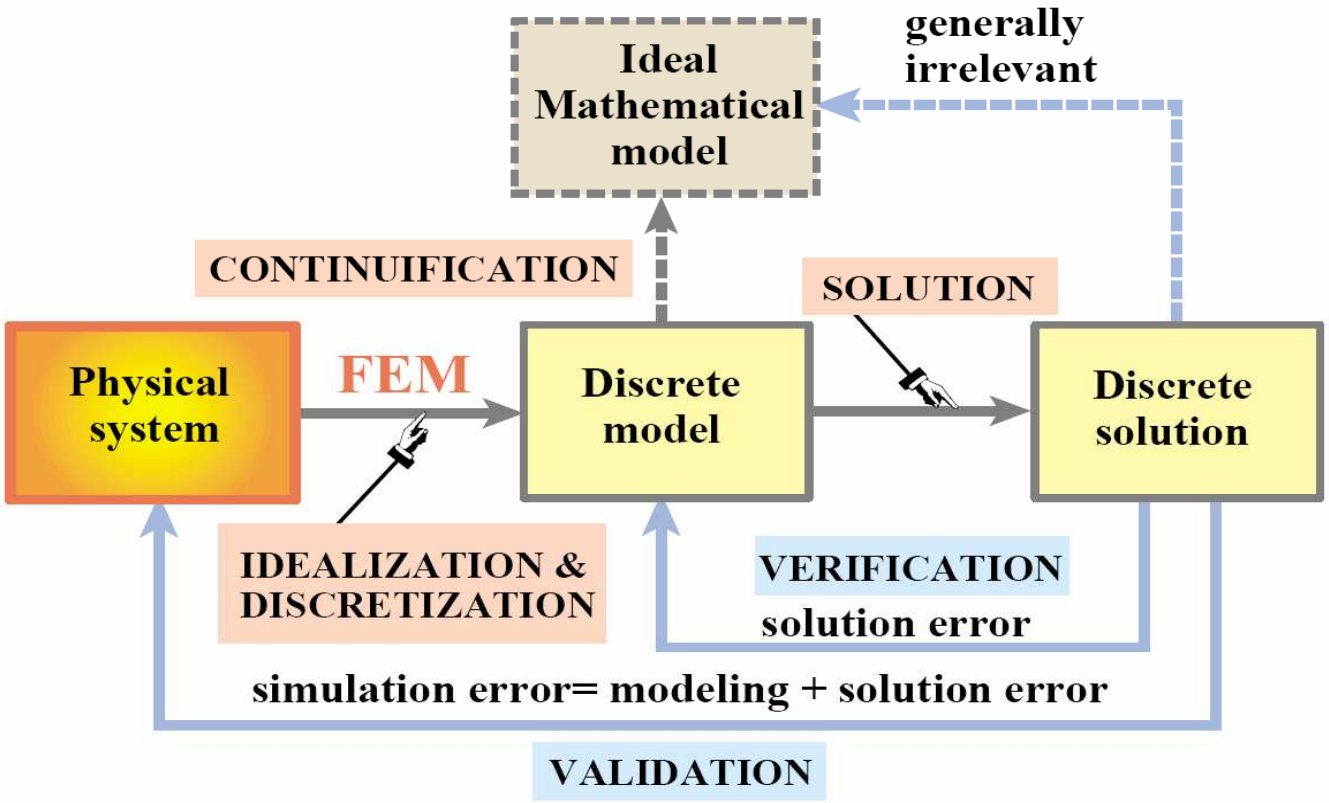
Mathematical *and* Discrete Model



Two Interpretations of FEM for Teaching

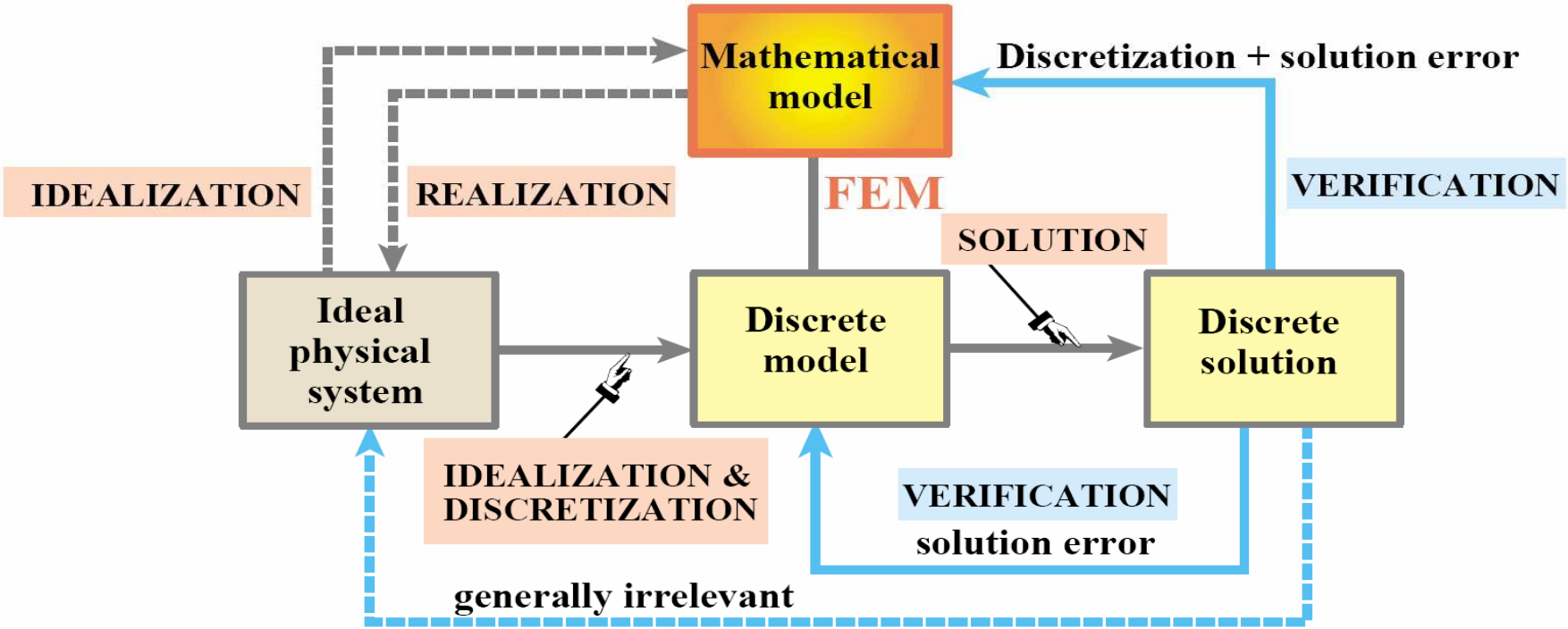
Physical	*Mathematical*
Breakdown of structural system into components (elements) and reconstruction by the assembly process	Numerical approximation of a Boundary Value Problem by Ritz-Galerkin discretization with functions of local support
Emphasized in Part I	Emphasized in Part II

FEM in Modeling and Simulation: Physical FEM

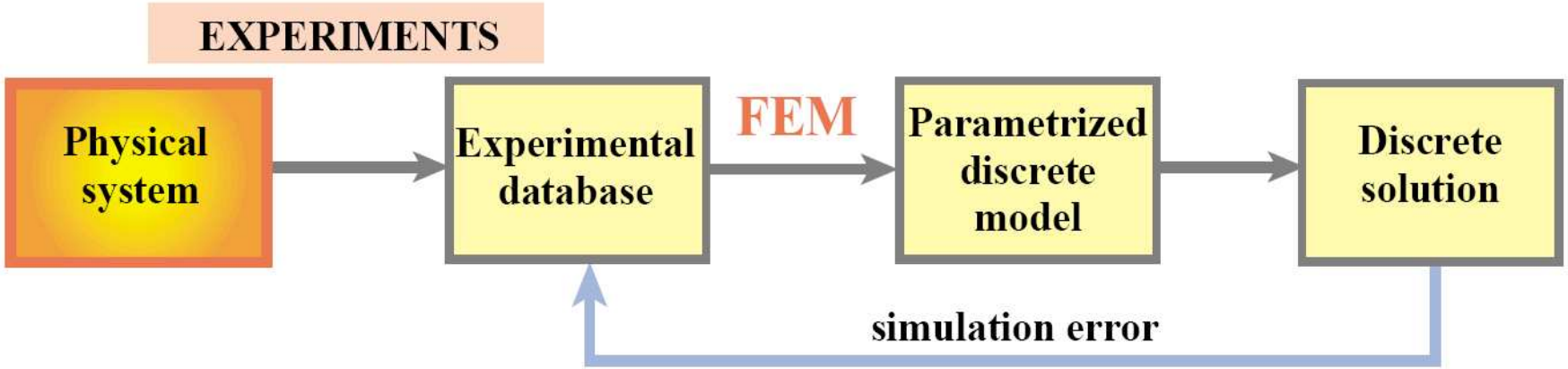


FEM: Overview

FEM in Modeling and Simulation: Mathematical FEM

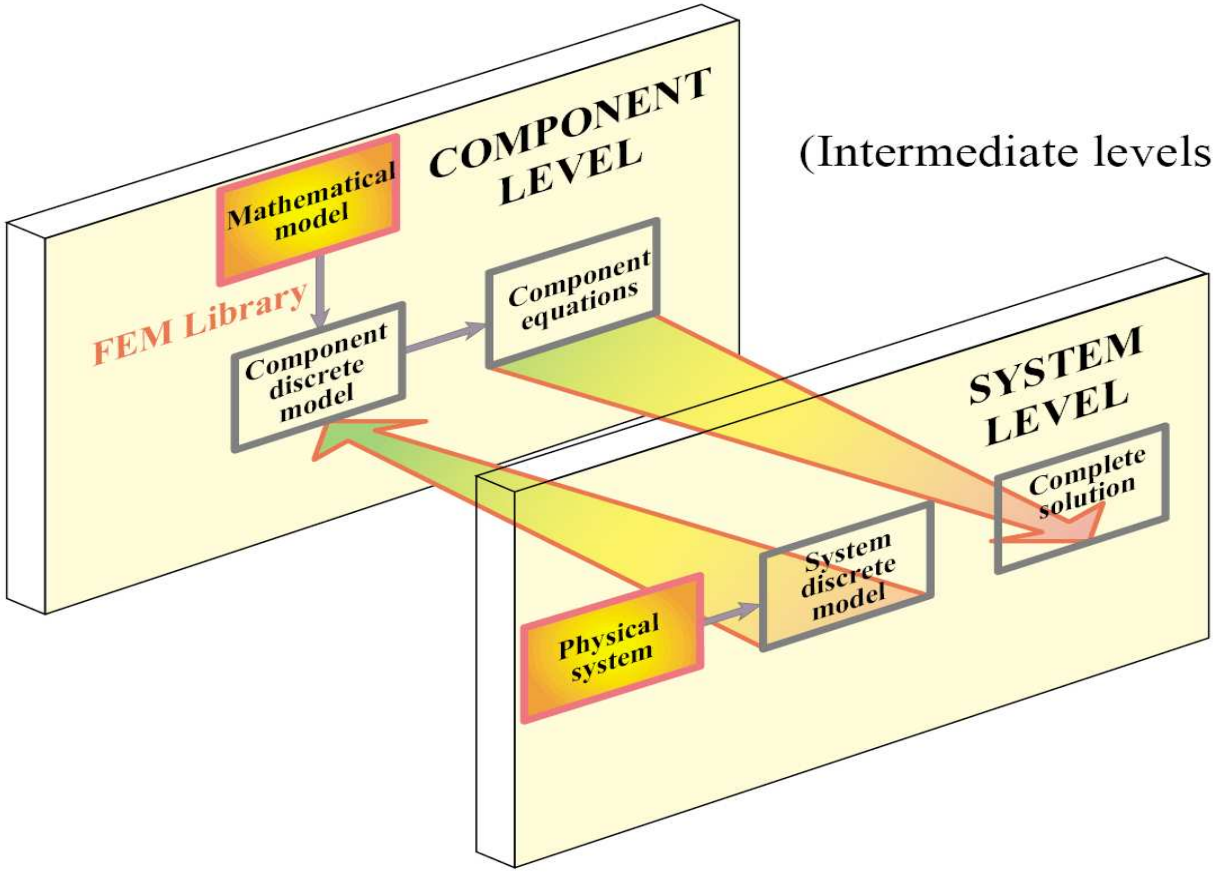


Model Updating in Physical FEM



FEM: Overview

Synergy Between Mathematical and Physical FEM



(Intermediate levels omitted)

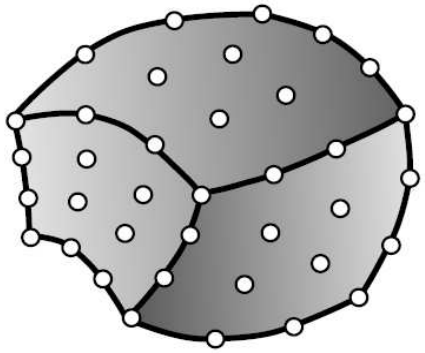
FEM: The Direct Stiffness Method

Importance: DSM is used by all major commercial FEM cod

A democratic method, works the same no matter what the element:



Bar (truss member) element,
2 nodes, 4 DOFs

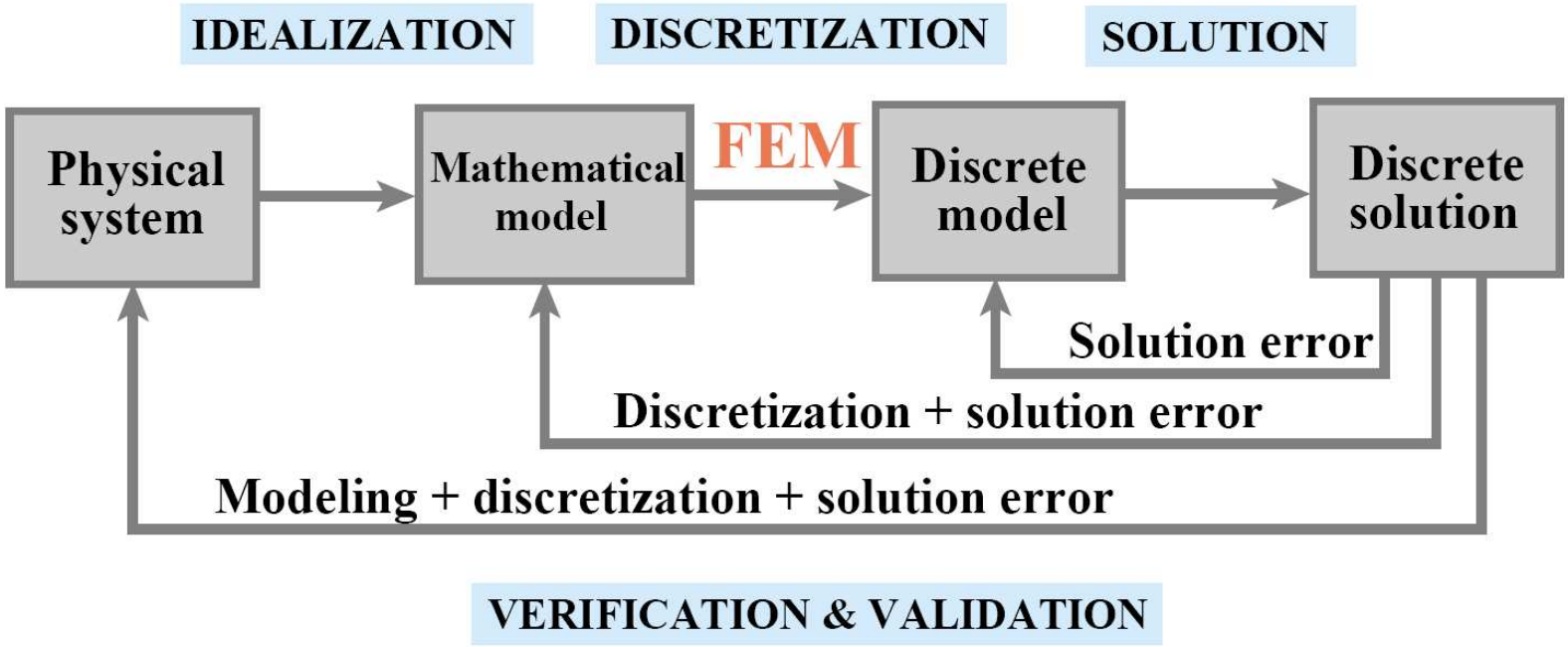


Tricubic brick element,
64 nodes, 192 DOFs

Obvious decision: use the truss to teach the DSM

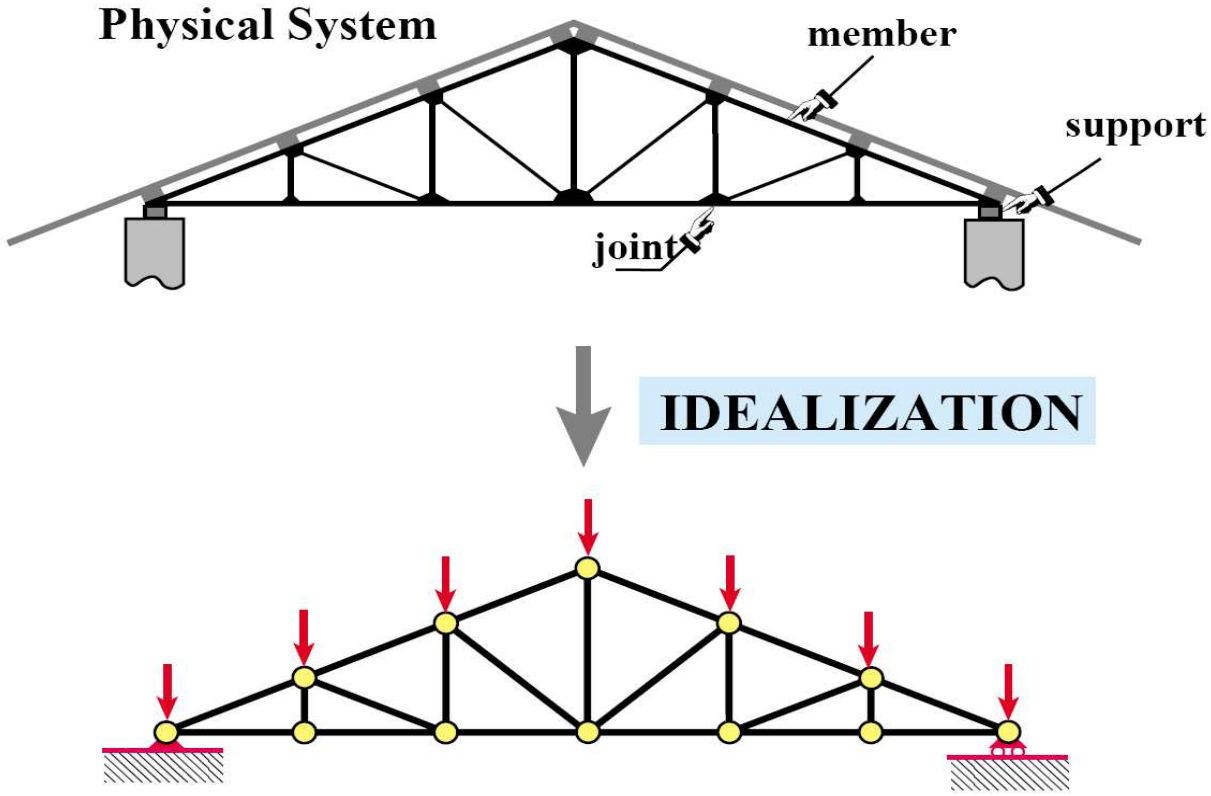
FEM: The Direct Stiffness Method

Model Based Simulation
(a simplification of diagrams of Chapter 1)



FEM: The Direct Stiffness Method

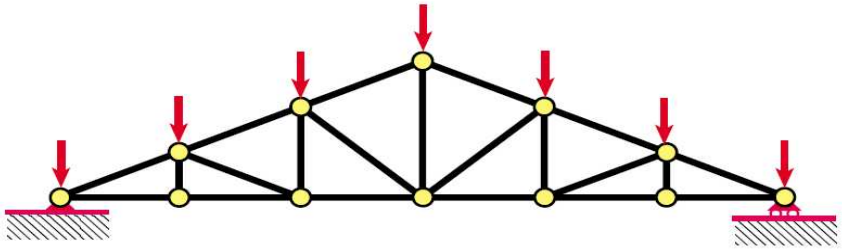
Idealization Process



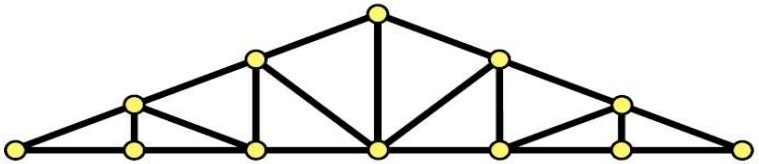
FEM: The Direct Stiffness Method

DSM: Breakdown Steps

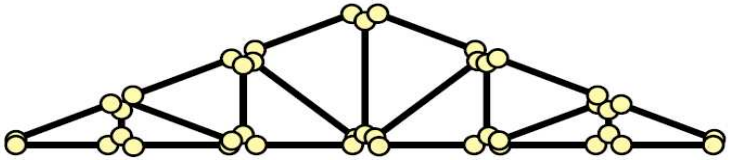
FEM model:



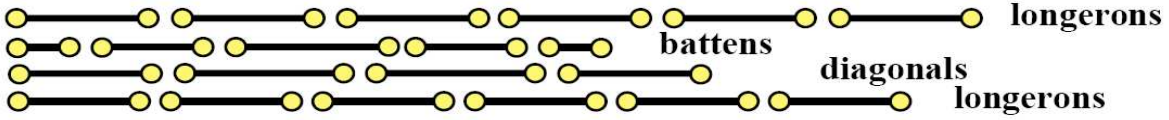
Remove loads & supports:



Disassemble:



Localize:



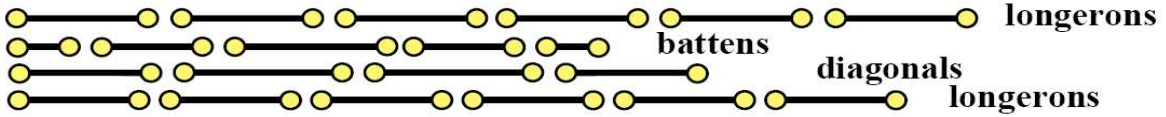
Generic element:



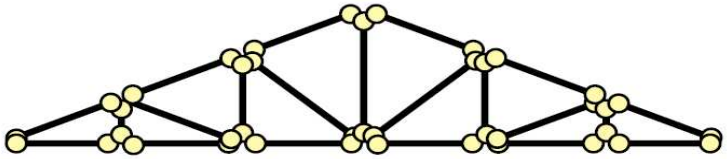
FEM: The Direct Stiffness Method

DSM: Assembly & Solution Steps

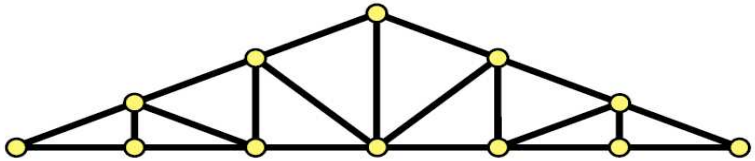
Form elements:



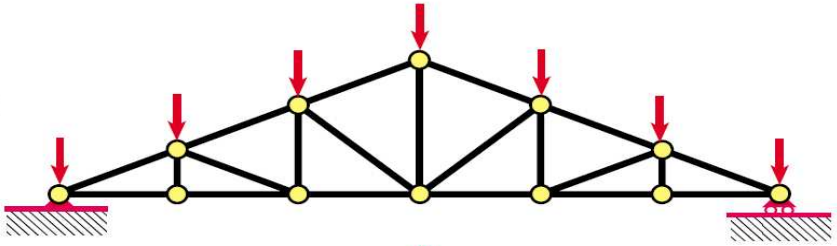
Globalize:



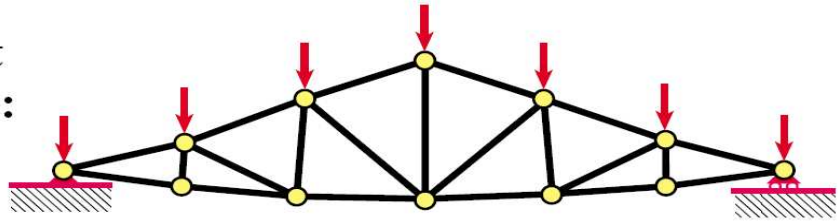
Merge:



Apply loads and supports:



Solve for joint displacements:



FEM: The Direct Stiffness Method

The Direct Stiffness Method (DSM) Steps
Starting with: Idealization

Breakdown (Chapter 2)

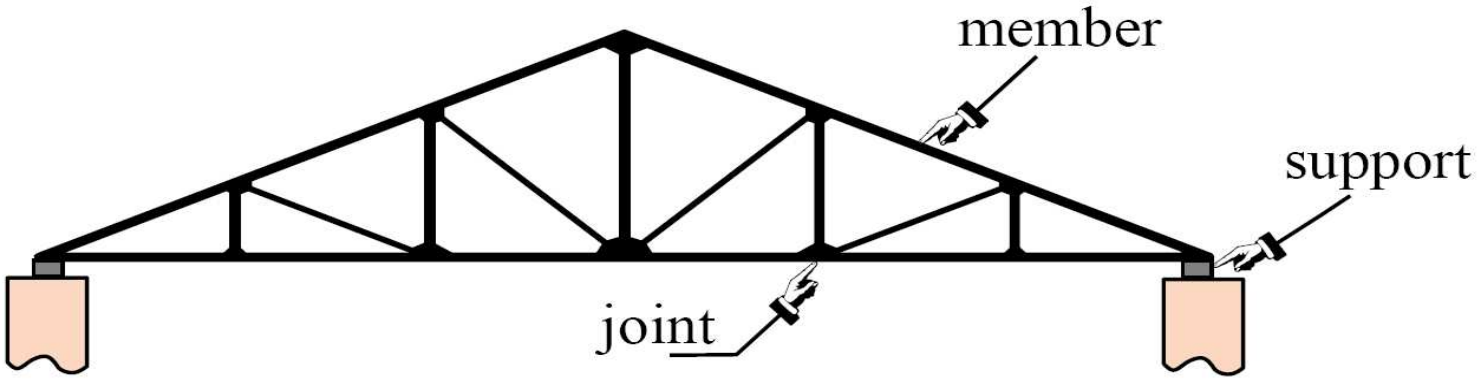
- Disconnection
- Localization
- Member (Element) Formation

Assembly & Solution (Chapter 3)

- Globalization
- Merge
- Application of BCs
- Solution
- Recovery of Derived Quantities

FEM: The Direct Stiffness Method

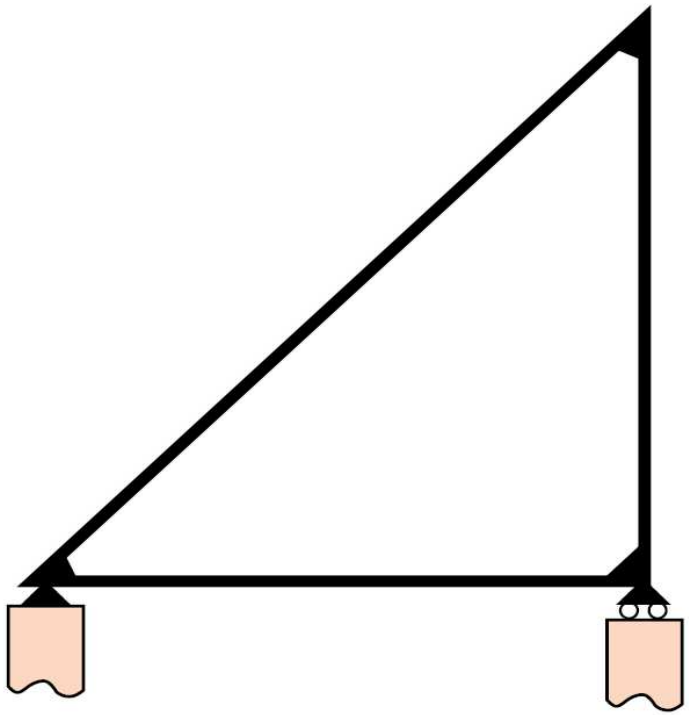
A Physical Plane Truss



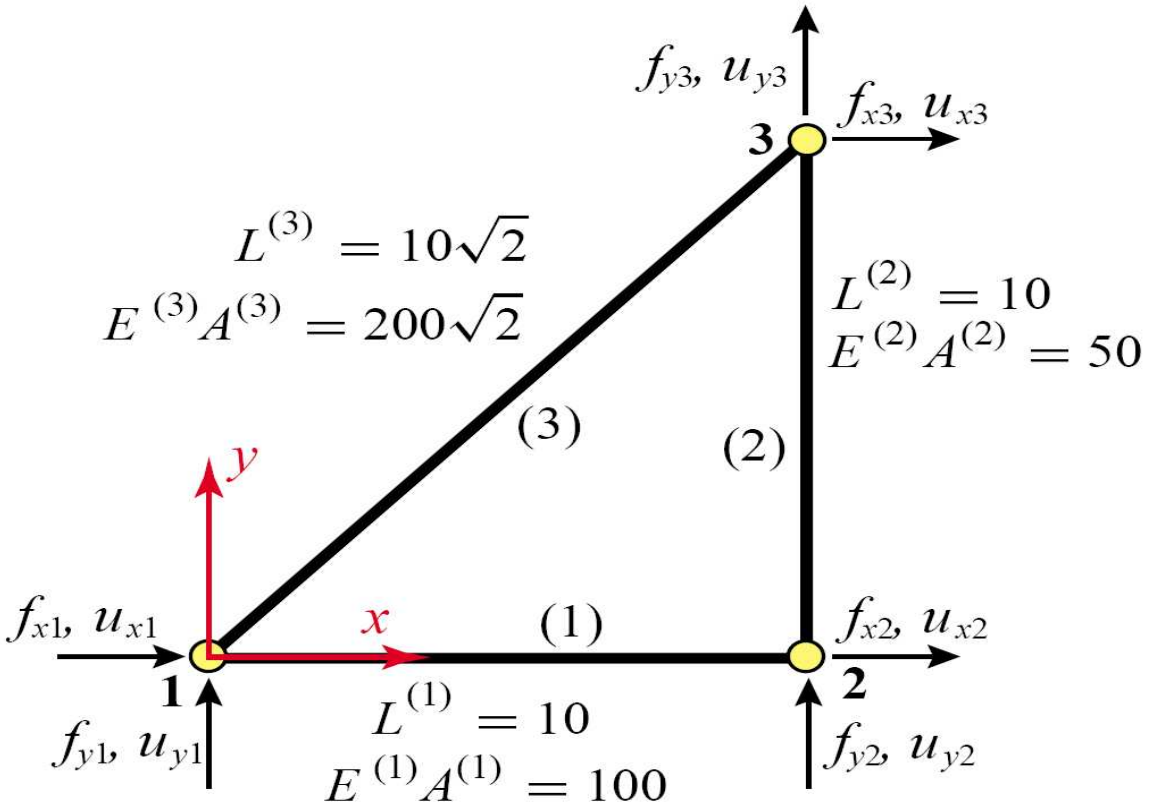
Too complicated to do by hand. We will use a simpler one to illustrate DSM steps

FEM: The Direct Stiffness Method

The Example Truss: Physical Model
(Loads not shown)



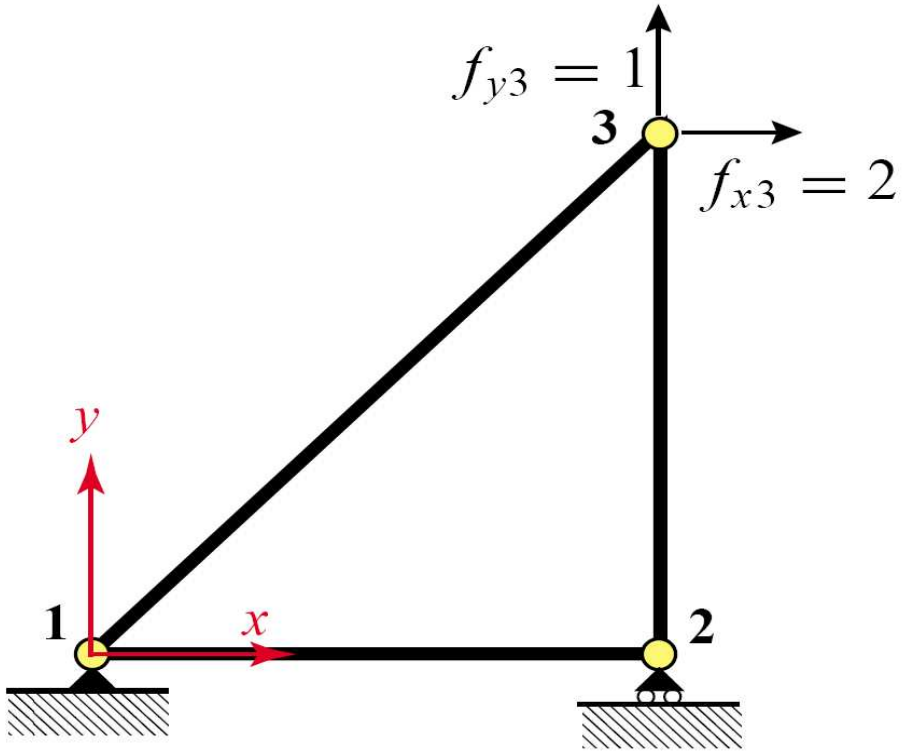
The Example Truss - FEM Model: Nodes, Elements and DOFs



The Example Truss - FEM Model BCs:

Applied Loads and Supports

Saved for Last



FEM: The Direct Stiffness Method

Master (Global) Stiffness Equations

$$\mathbf{f} = \begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Linear structure:

$$\begin{bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \end{bmatrix} = \begin{bmatrix} K_{x1x1} & K_{x1y1} & K_{x1x2} & K_{x1y2} & K_{x1x3} & K_{x1y3} \\ K_{y1x1} & K_{y1y1} & K_{y1x2} & K_{y1y2} & K_{y1x3} & K_{y1y3} \\ K_{x2x1} & K_{x2y1} & K_{x2x2} & K_{x2y2} & K_{x2x3} & K_{x2y3} \\ K_{y2x1} & K_{y2y1} & K_{y2x2} & K_{y2y2} & K_{y2x3} & K_{y2y3} \\ K_{x3x1} & K_{x3y1} & K_{x3x2} & K_{x3y2} & K_{x3x3} & K_{x3y3} \\ K_{y3x1} & K_{y3y1} & K_{y3x2} & K_{y3y2} & K_{y3x3} & K_{y3y3} \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix}$$

Nodal forces — **Master stiffness matrix** — **Nodal displacements**

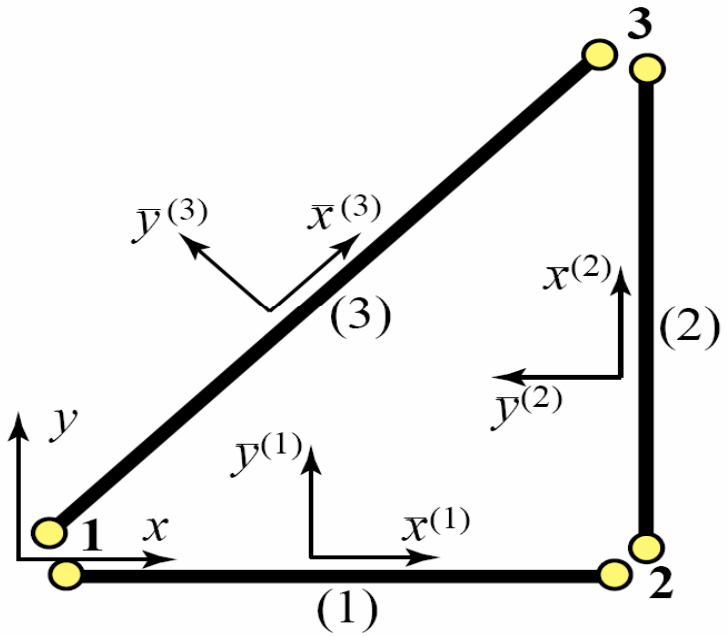
or $\mathbf{f} = \mathbf{K}\,\mathbf{u}$

Member (Element) Stiffness Equations

$$\bar{\mathbf{f}} = \bar{\mathbf{K}}\,\bar{\mathbf{u}}$$

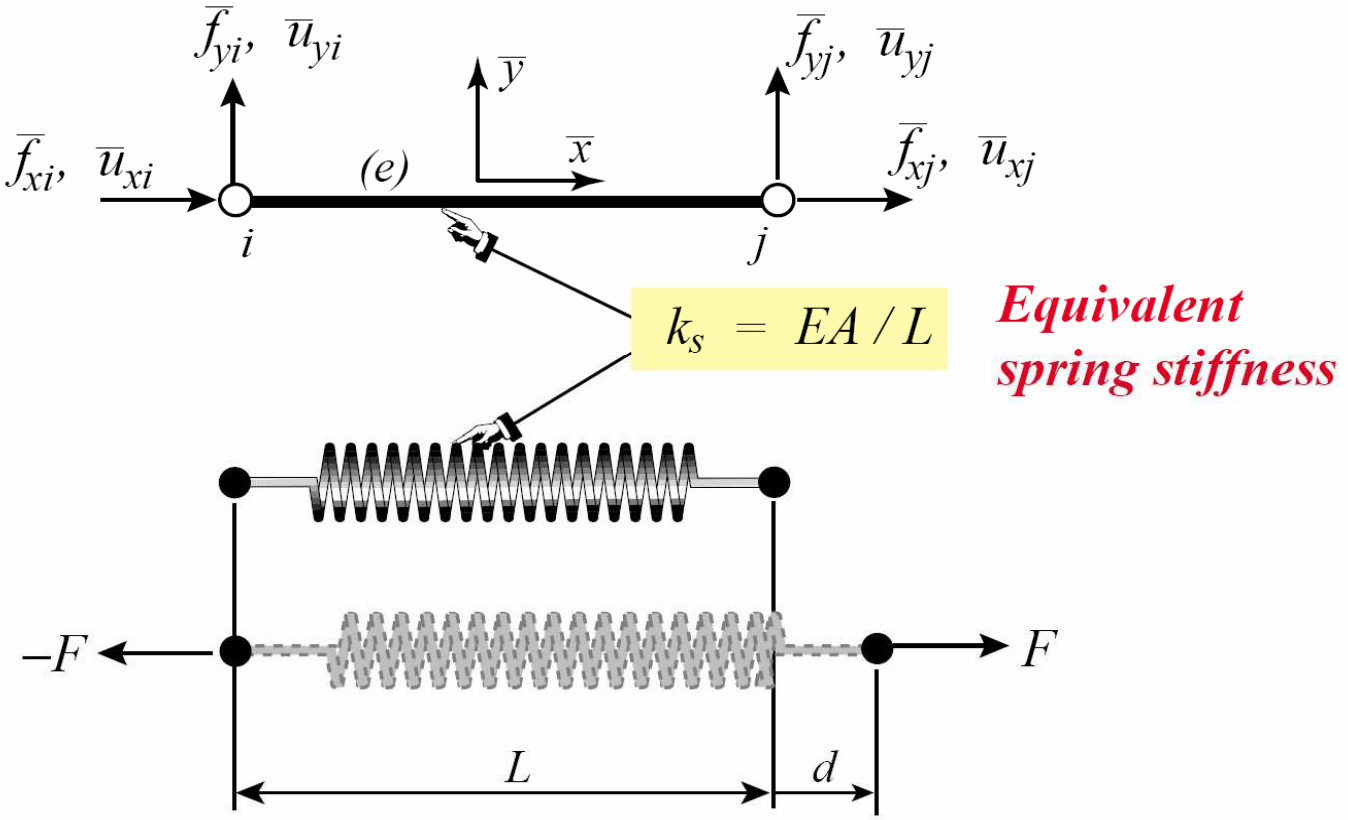
$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \begin{bmatrix} \bar{K}_{xixi} & \bar{K}_{xiyi} & \bar{K}_{xixj} & \bar{K}_{xiyj} \\ \bar{K}_{yixi} & \bar{K}_{yiyi} & \bar{K}_{yixj} & \bar{K}_{yiyj} \\ \bar{K}_{xjxi} & \bar{K}_{xjyi} & \bar{K}_{xjxj} & \bar{K}_{xjyj} \\ \bar{K}_{yjxi} & \bar{K}_{yjyi} & \bar{K}_{yjxj} & \bar{K}_{yjyj} \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

First Two Breakdown Steps: Disconnection and Localization



These steps are conceptual (not actually programmed)

The 2-Node Truss (Bar) Element



Truss (Bar) Element Formulation by Mechanics of Materials (MoM)

$$F = k_s d = \frac{EA}{L} d, \quad F = \bar{f}_{xj} = -\bar{f}_{xi}, \quad d = \bar{u}_{xj} - \bar{u}_{xi}$$

Exercise 2.3

$$\begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix}$$

Element stiffness equations in local coordinates

from which

$$\overline{\mathbf{K}} = \frac{EA}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element stiffness matrix in local coordinates

FEM: The Direct Stiffness Method

Where We Are So Far in the DSM

we are done with this ...

Breakdown (Chapter 2)

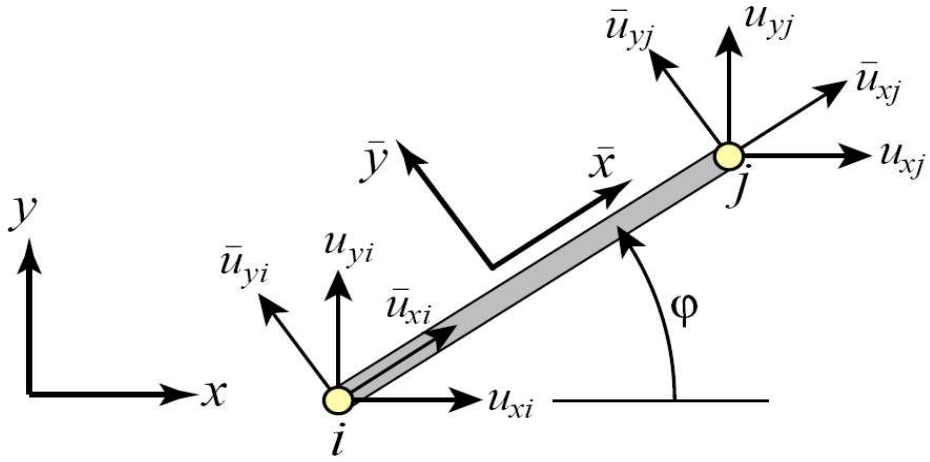
- Disconnection
- Localization
- Member (Element) Formation

we finish Chapter 2 with

Assembly & Solution (Chapter 3)

- Globalization
- Merge
- Application of BCs
- Solution
- Recovery of Derived Quantities

Globalization: Displacement Transformation



Node displacements transform as

$$\bar{u}_{xi} = u_{xi}c + u_{yi}s, \qquad \bar{u}_{yi} = -u_{xi}s + u_{yi}c\gamma$$
$$\bar{u}_{xj} = u_{xj}c + u_{yj}s, \qquad \bar{u}_{yj} = -u_{xj}s + u_{yj}c\gamma$$

in which $c = \cos\varphi \quad s = \sin\varphi$

Displacement Transformation (cont'd)

In matrix form

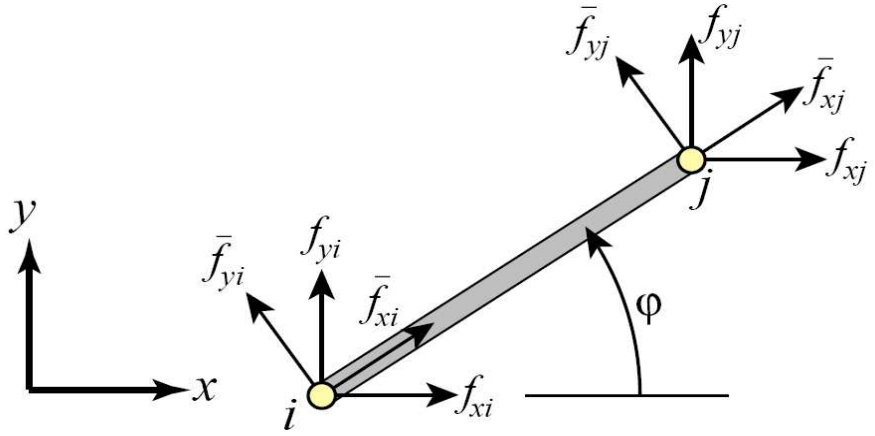
$$\begin{bmatrix} \bar{u}_{xi} \\ \bar{u}_{yi} \\ \bar{u}_{xj} \\ \bar{u}_{yj} \end{bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{bmatrix}$$

or

$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e$$

Note:
global on RHS,
local on LHS

Globalization: Force Transformation



Node forces transform as

$$\begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{bmatrix} = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix} \begin{bmatrix} \bar{f}_{xi} \\ \bar{f}_{yi} \\ \bar{f}_{xj} \\ \bar{f}_{yj} \end{bmatrix}$$

Note: global on LHS, local on RHS

or

$$\mathbf{f}^e = (\mathbf{T}^e)^T \, \bar{\mathbf{f}}^e$$

Globalization: Congruential Transformation of Element Stiffness Matrices

$$\bar{\mathbf{K}}^e \; \mathbf{u}^e = \bar{\mathbf{f}}^e$$

$$\bar{\mathbf{u}}^e = \mathbf{T}^e \mathbf{u}^e \qquad \mathbf{f}^e = (\mathbf{T}^e)^T \; \bar{\mathbf{f}}^e$$

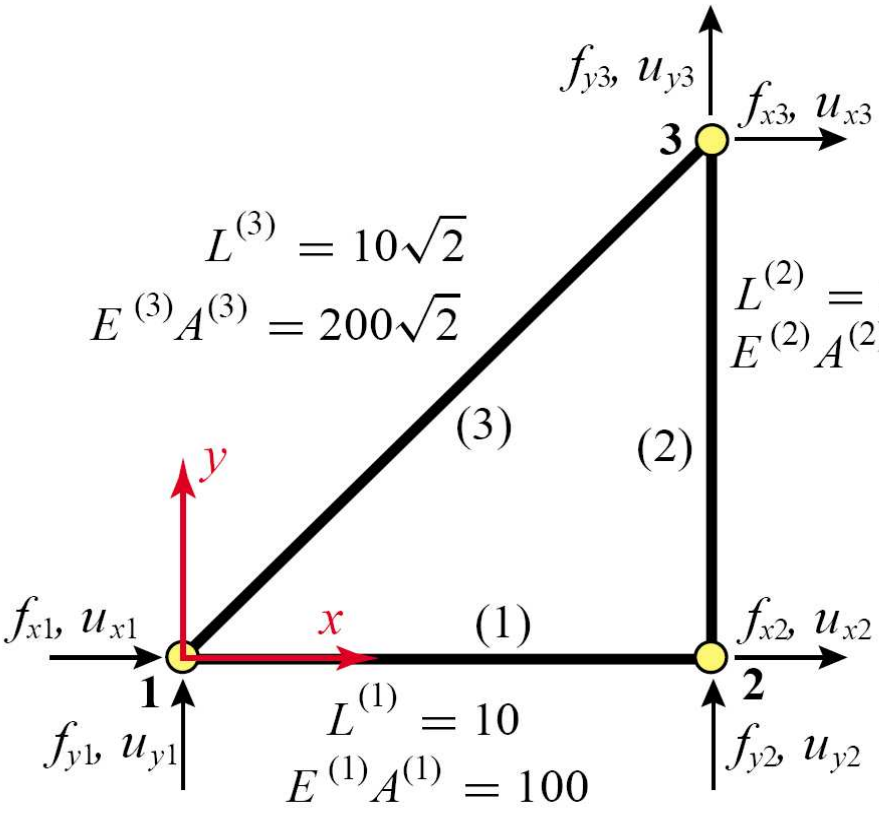
$$\mathbf{K}^e = (\mathbf{T}^e)^T \; \bar{\mathbf{K}}^e \; \mathbf{T}^e$$

Exercise 2.8

$$\mathbf{K}^e = \frac{E^e A^e}{L^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

FEM: The Direct Stiffness Method

The Example Truss - FEM Model (Recalled for Convenience)



Insert the geometric & physical properties of this model into the globalized member stiffness equations

FEM: The Direct Stiffness Method

We Obtain the Globalized Element Stiffness Equations of the Example Truss

$$\begin{bmatrix} f_{x1}^{(1)} \\ f_{y1}^{(1)} \\ f_{x2}^{(1)} \\ f_{y2}^{(1)} \end{bmatrix} = 10 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{x1}^{(1)} \\ u_{y1}^{(1)} \\ u_{x2}^{(1)} \\ u_{y2}^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} f_{x2}^{(2)} \\ f_{y2}^{(2)} \\ f_{x3}^{(2)} \\ f_{y3}^{(2)} \end{bmatrix} = 5 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_{x2}^{(2)} \\ u_{y2}^{(2)} \\ u_{x3}^{(2)} \\ u_{y3}^{(2)} \end{bmatrix}$$

In the next class we will put these to good use

$$\begin{bmatrix} f_{x1}^{(3)} \\ f_{y1}^{(3)} \\ f_{x3}^{(3)} \\ f_{y3}^{(3)} \end{bmatrix} = 20 \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_{x1}^{(3)} \\ u_{y1}^{(3)} \\ u_{x3}^{(3)} \\ u_{y3}^{(3)} \end{bmatrix}$$

FEM Terminology

degrees of freedom (abbrv: DOF)

state (primary) variables: displacements in mechanics

conjugate variables: forces in mechanics

stiffness matrix

master stiffness equations

$$\mathbf{K}\,\mathbf{u} = \mathbf{f}$$

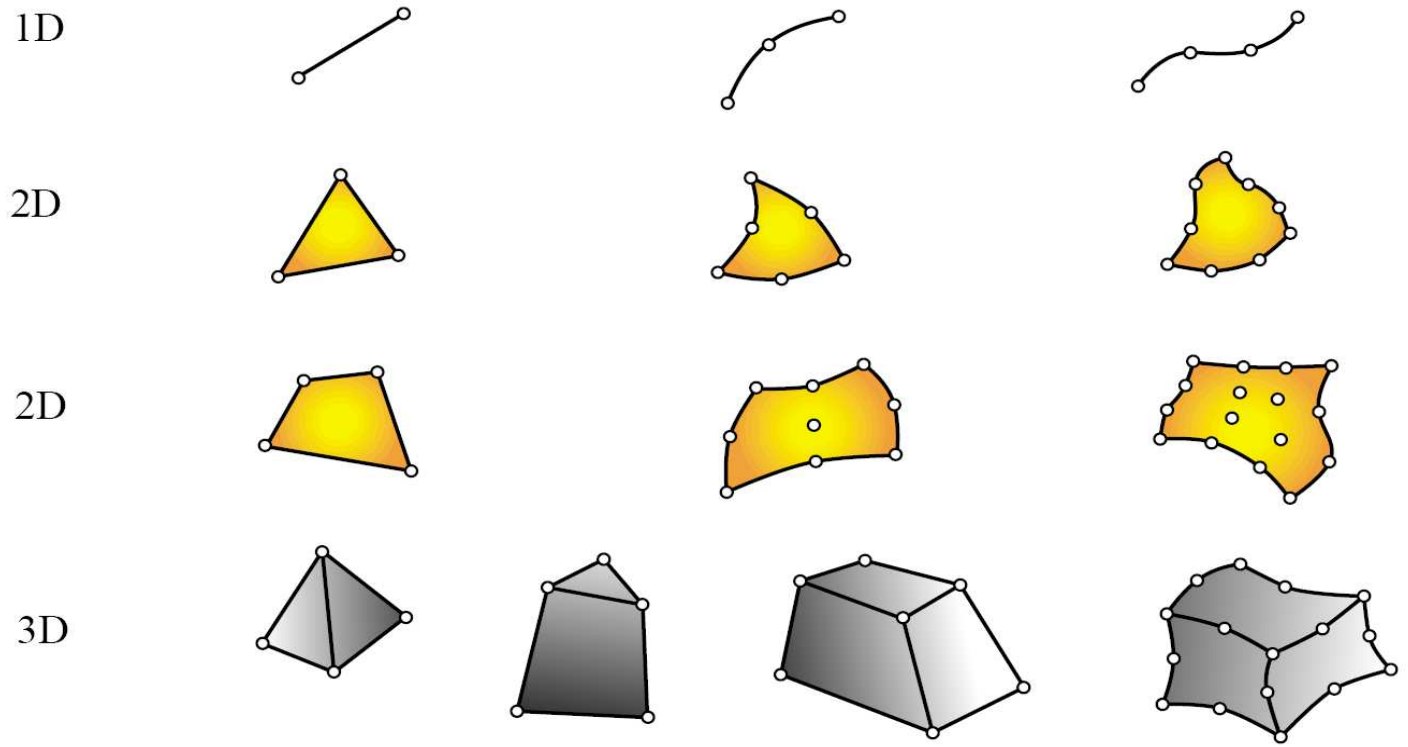
$$\mathbf{K}\,\mathbf{u} = \mathbf{f}_M + \mathbf{f}_I$$

Physical Significance of Vectors u and f in Miscellaneous FEM Applications

Application Problem	*State (DOF) vector* **u** *represents*	*Forcing vector* **f** *represents*
Structures and solid mechanics	Displacement	Mechanical force
Heat conduction	Temperature	Heat flux
Acoustic fluid	Displacement potential	Particle velocity
Potential flows	Pressure	Particle velocity
General flows	Velocity	Fluxes
Electrostatics	Electric potential	Charge density
Magnetostatics	Magnetic potential	Magnetic intensity

FEM Modeling: Introduction

Element Geometry Is Defined by Node Locations



Classification of Mechanical Finite Elements

Primitive Structural

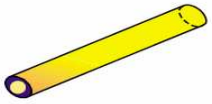
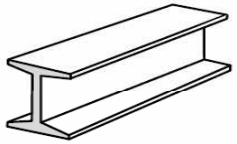
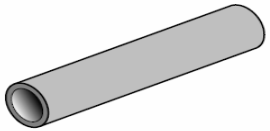
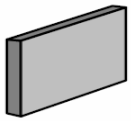
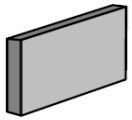
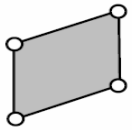
Continuum

Special

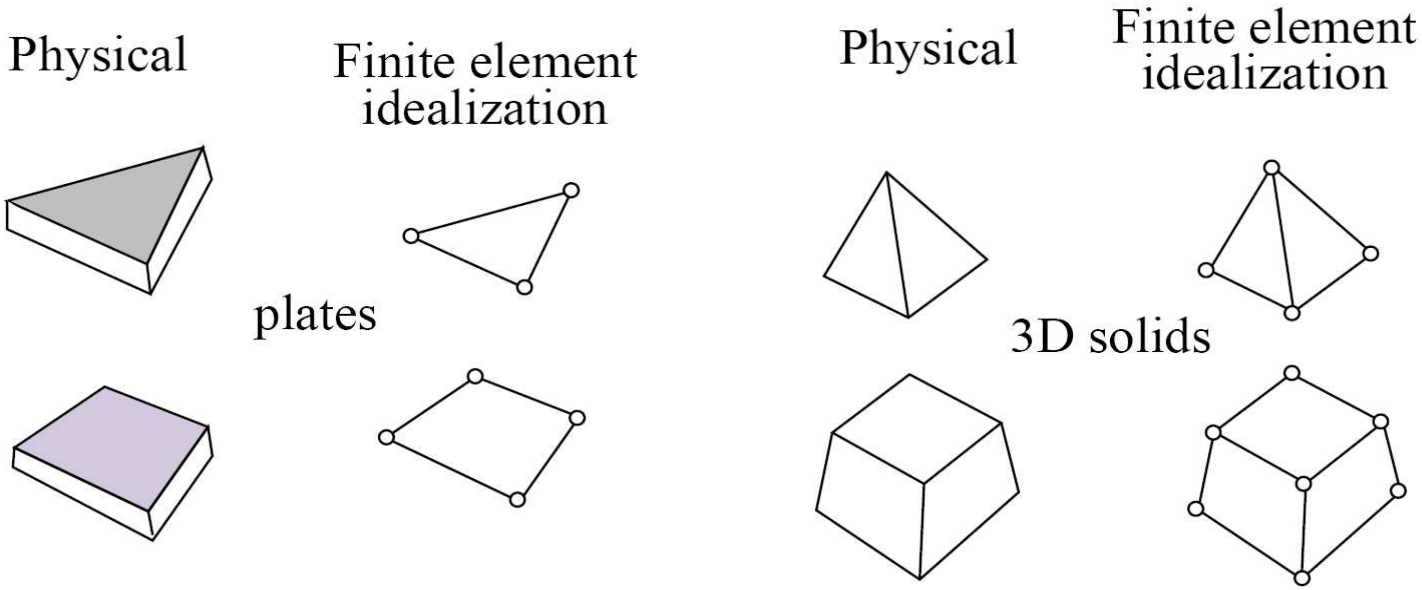
Macroelements } Superelements

Substructures }

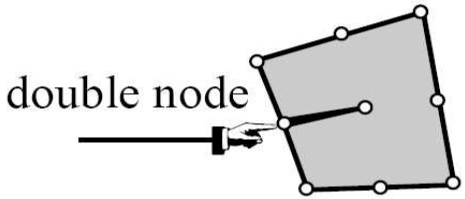
Primitive Structural Elements
(often built from MoM models)

Physical Structural Component	Mathematical Model Name	Finite Element Discretization
	bar	
	beam	
	tube, pipe	
	spar (web)	
	shear panel (2D version of above)	

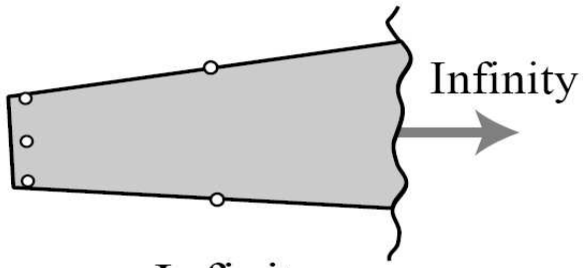
Continuum Elements



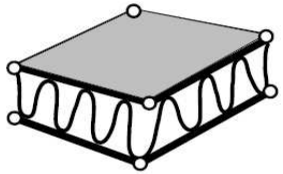
Special Elements



Crack element

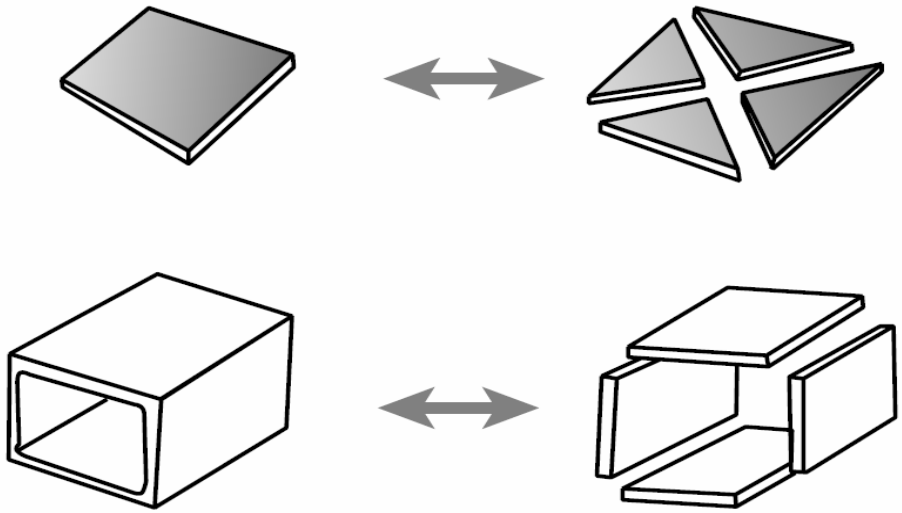


Infinite element

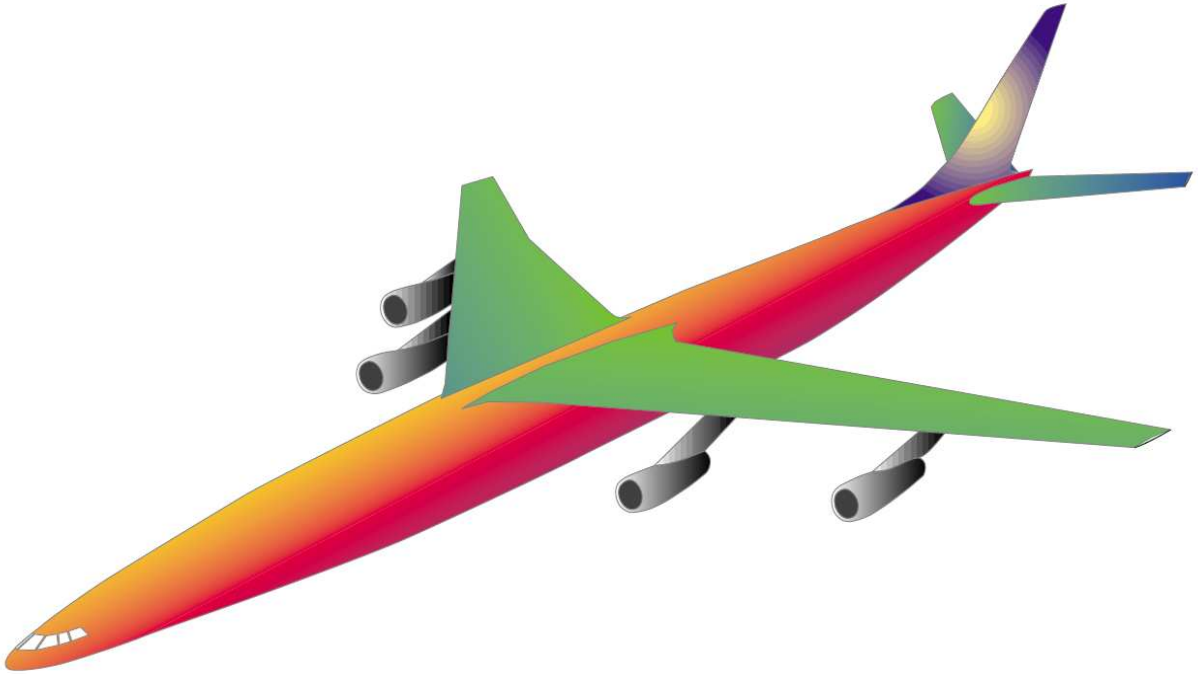


Honeycomb panel

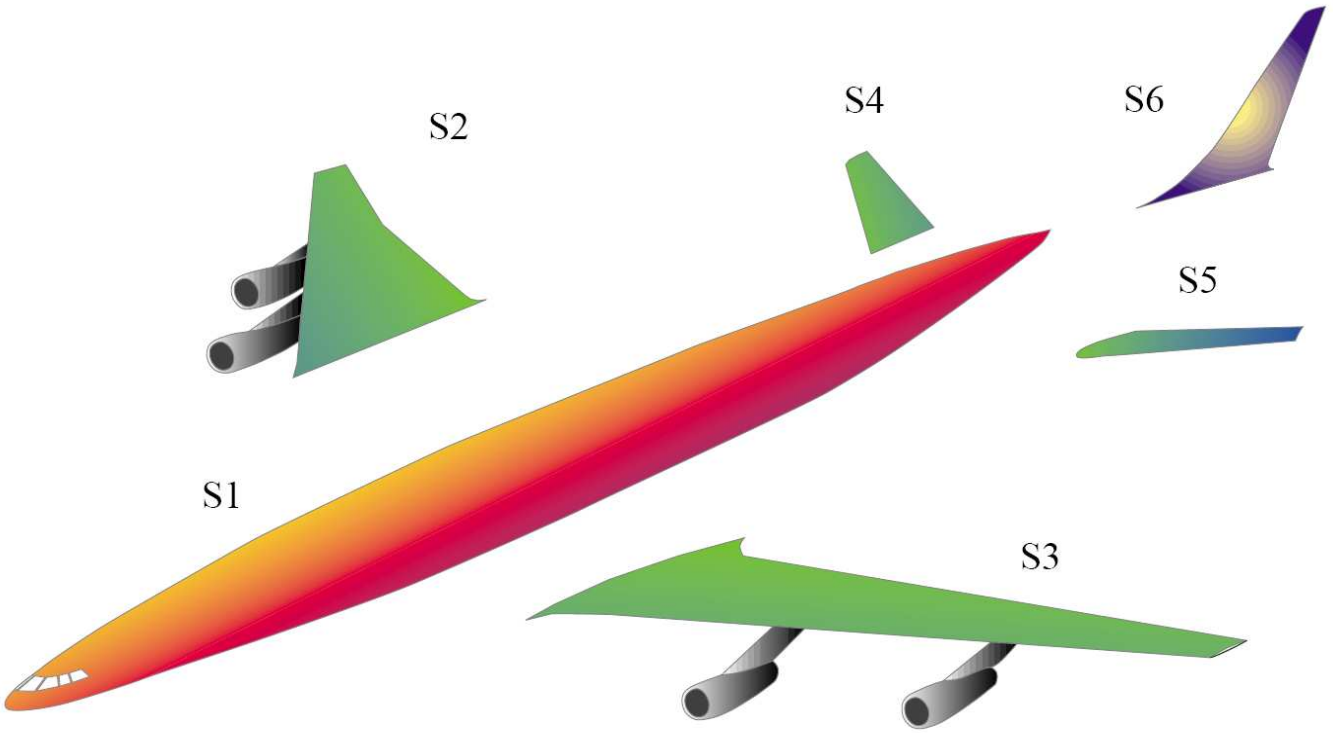
MacroElements



Substructures



Substructures (cont'd)



FEM Modeling: Introduction

Boundary Conditions (BCs)

The most difficult topic for FEM program users ("the devil hides in the boundary")

Two types { **Essential**, **Natural**

Boundary Conditions
Essential vs. Natural

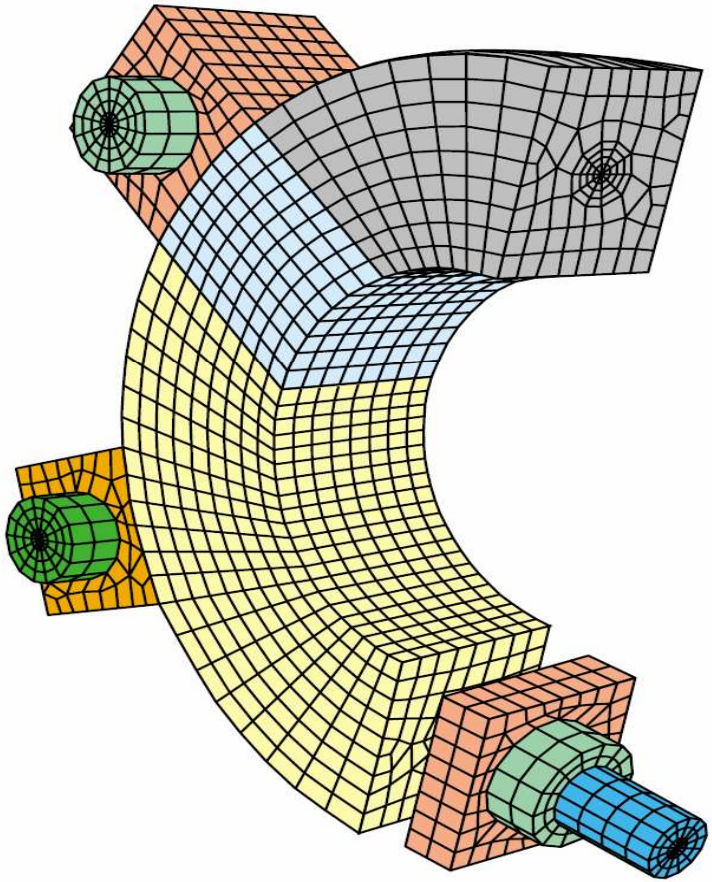
Recipe:

1. **If a BC involves one or more DOF in a *direct* way, it is *essential* and goes to the Left Hand Side (LHS) of Ku = f**

2. **Otherwise it is *natural* and goes to the Right Hand Side (RHS) of Ku = f**

FEM Modeling: Introduction

Examples of Structural Models: Machine Component (Mech. Engrg)



FEM Modeling: Mesh, Loads and BCs

General FEM Modeling Rules

- **Use the simplest elements that will do the job**
- ***Never, never, never*** **use complicated or special elements unless you are absolutely sure of what you are doing**
- **Use the coarsest mesh that will capture the dominant behavior of the physical model, particularly in *design* situations**

3 word summary: *Keep It Simple*

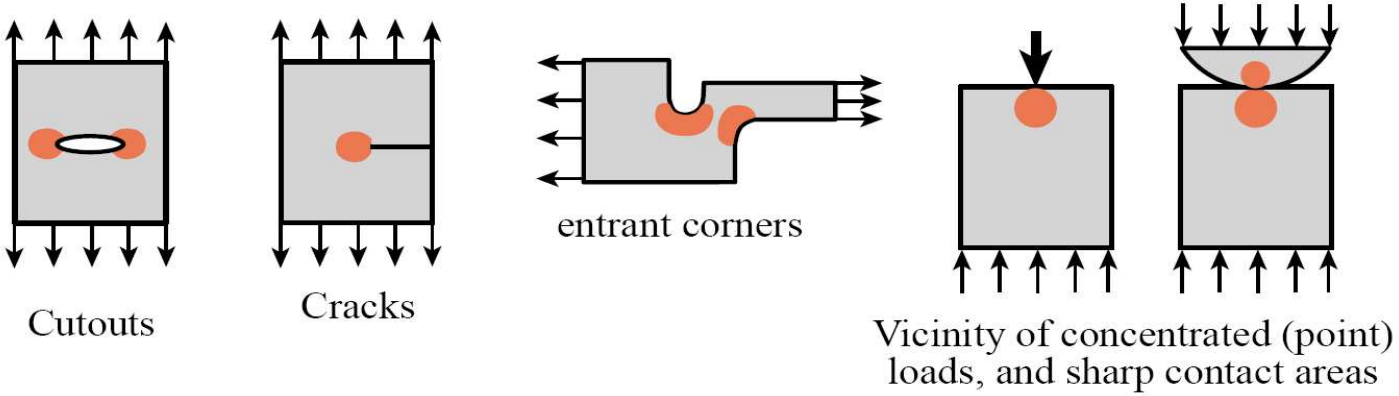
Another Justification for Simplicity

In product design situations
several FEM models of increasing refinement
will be set up as design evolves

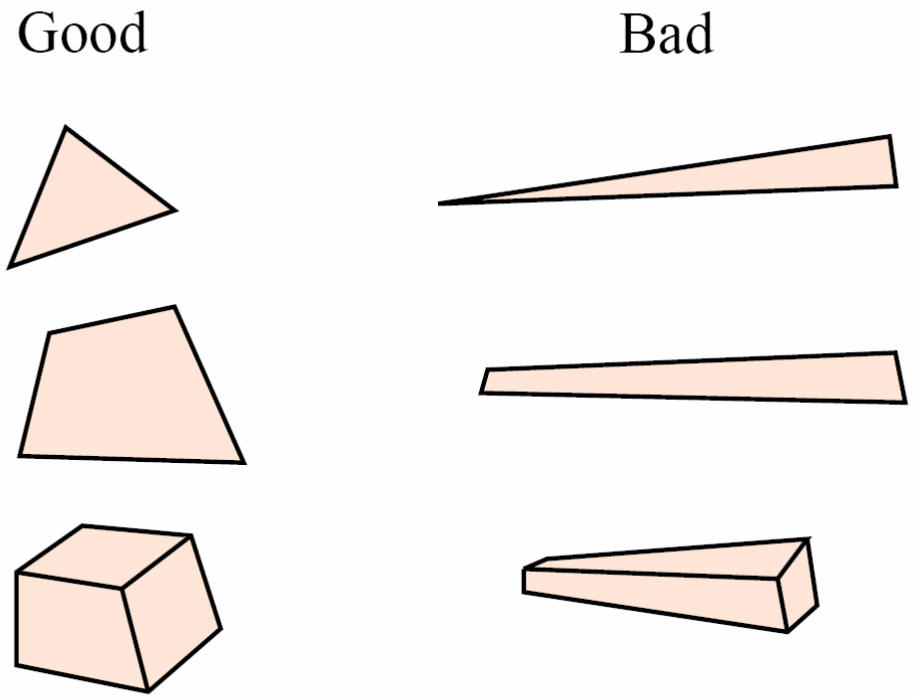
Ergo, do not overkill at the beginning

FEM Modeling: Mesh, Loads and BCs

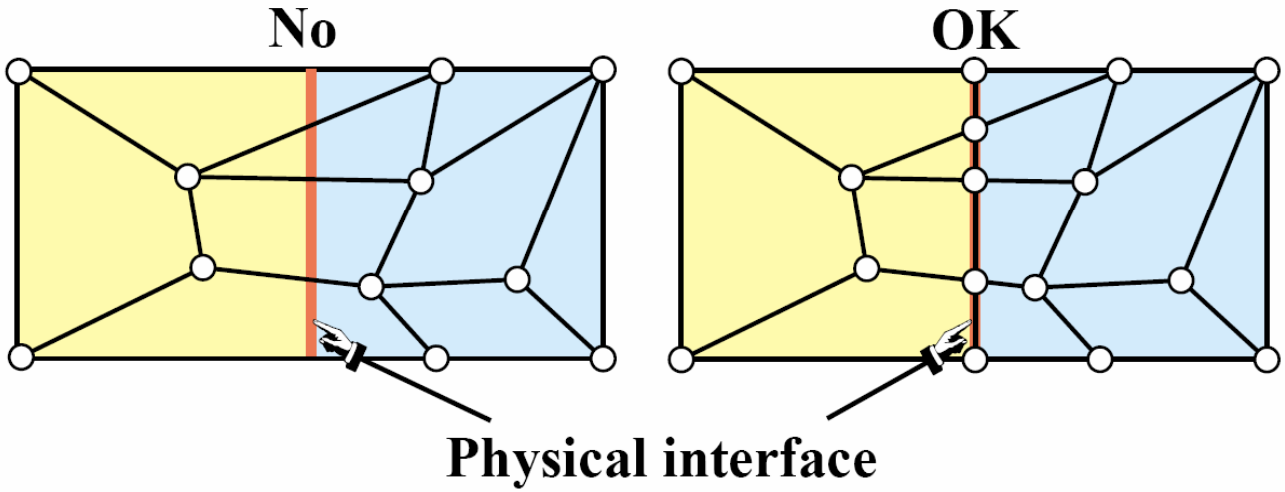
Where Finer Meshes Should be Used



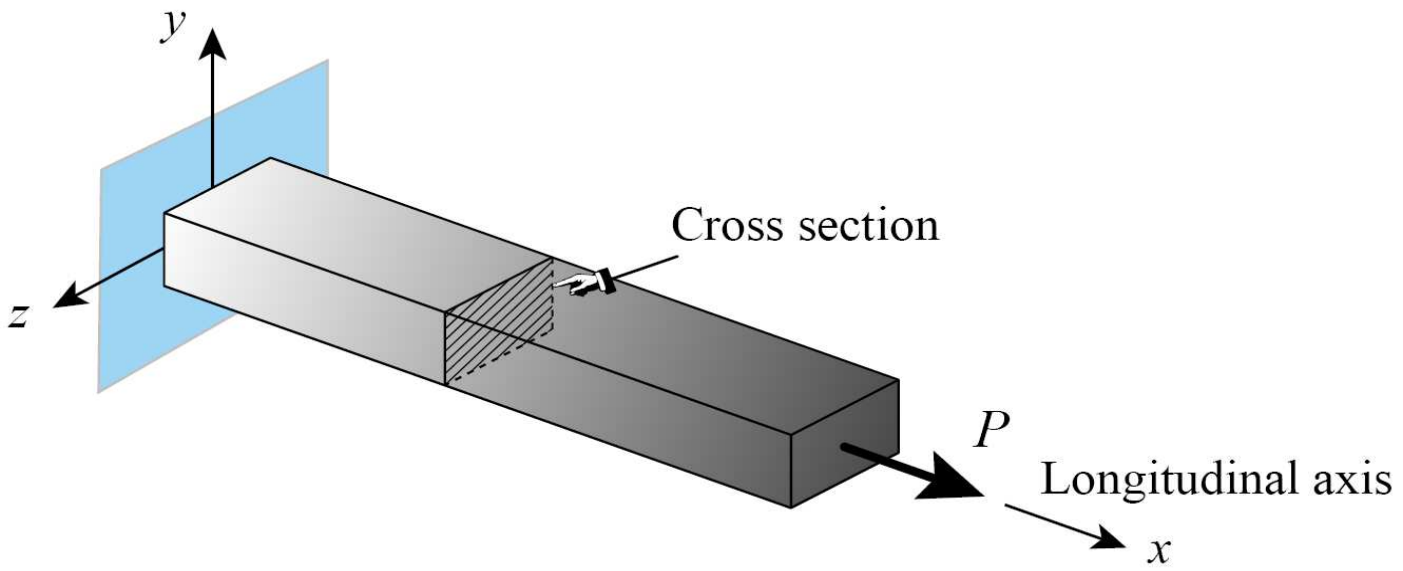
Avoid 2D/3D Elements of Bad Aspect Ratio



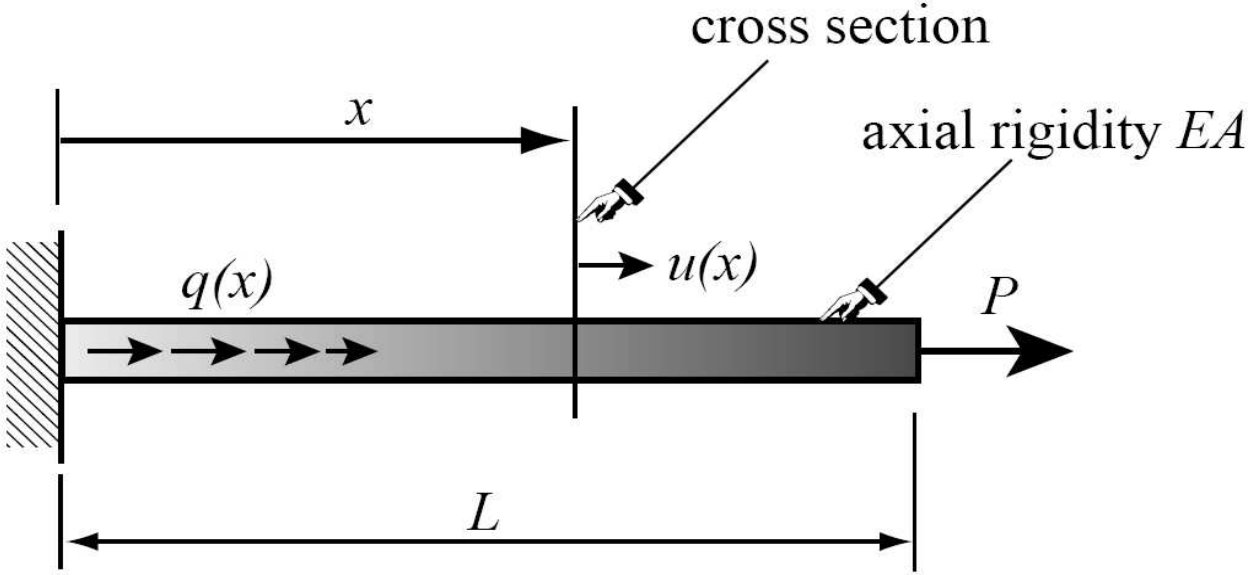
Elements Must Not Cross Interfaces



Bar Member - Variational Derivation



Bar Member (cont'd)



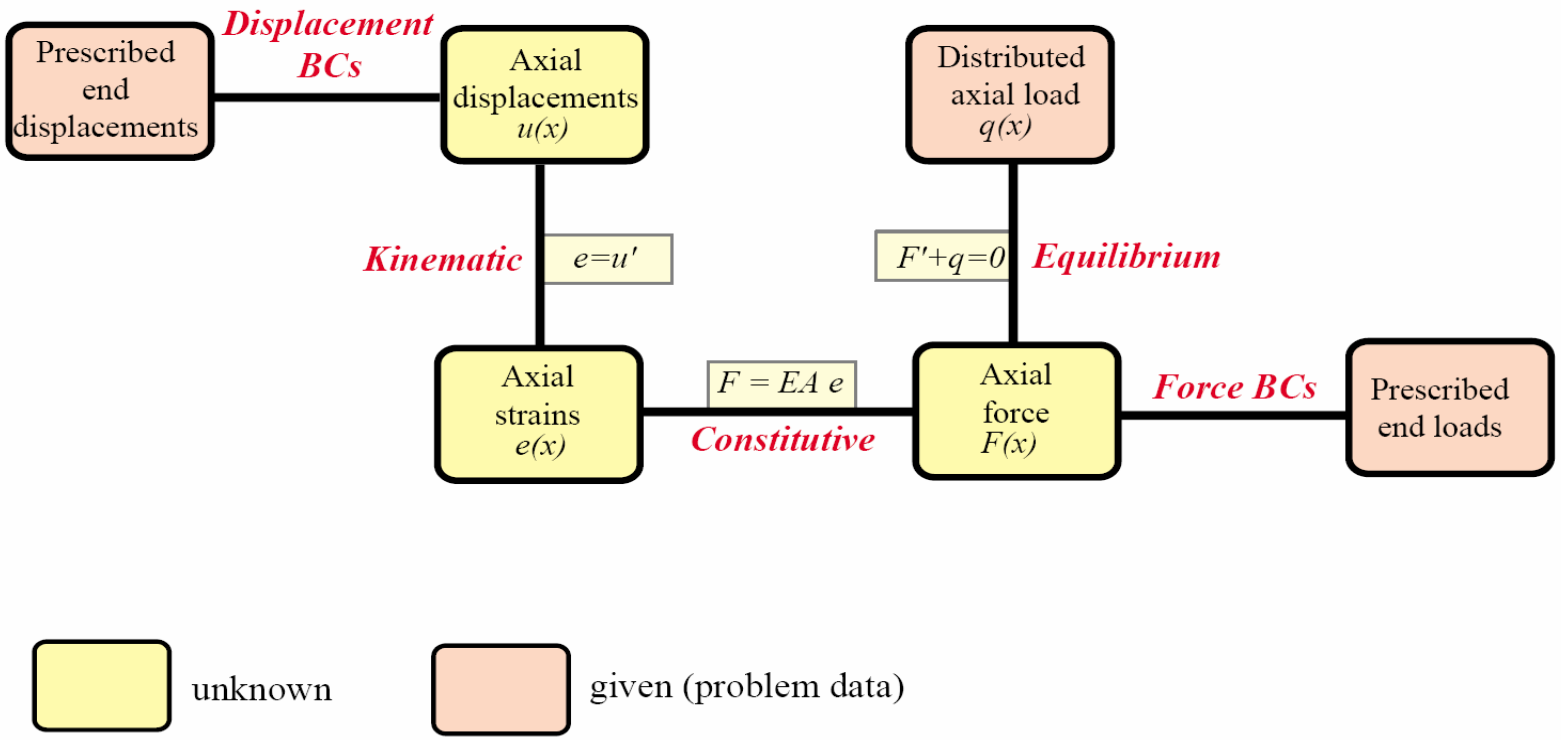
The Bar Revisited - Notation

Quantity	*Meaning*
x	Longitudinal bar axis *
$(.)'$	$d(.)/dx$
$u(x)$	Axial displacement
$q(x)$	Distributed axial force, given per unit of bar length
L	Total length of bar member
E	Elastic modulus
A	Cross section area, may vary with x
EA	Axial rigidity
$e = du/dx = u'$	Infinitesimal axial strain
$\sigma = E\, e = E\, u'$	Axial stress
$F = A\, \sigma = EA\, e = EA\, u'$	Internal axial force
P	Prescribed end load

* x is used in this Chapter instead of $\bar{x}$ (as in Chapters 2-3) to simplify the notation

FEM: Variational Formulation of Bar Element

Tonti Diagram of Governing Equations



FEM: Variational Formulation of Bar Element

Potential Energy of the Bar Member

(before discretization)

Internal energy (= strain energy)

$$U = \frac{1}{2}\int_0^L Fe\,dx = \frac{1}{2}\int_0^L (EAu')u'\,dx = \frac{1}{2}\int_0^L u'EAu'\,dx$$

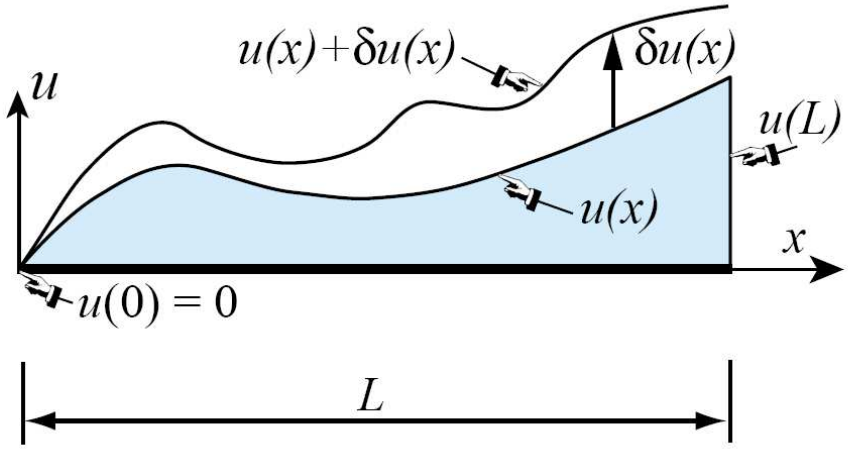
External work

$$W = \int_0^L qu\,dx$$

Total potential energy

$$\Pi = U - W$$

Concept of Kinematically Admissible Variation



$\delta u(x)$ is ***kinematically admissible*** if $u(x)$ and $u(x) + \delta u(x)$

(i) are ***continuous*** over bar length, i.e. $u(x) \in C_0$ in $x \in [0, L]$.

(ii) ***satisfy exactly displacement BC***; in the figure, $u(0) = 0$

The Minimum Potential Energy (MPE) Principle

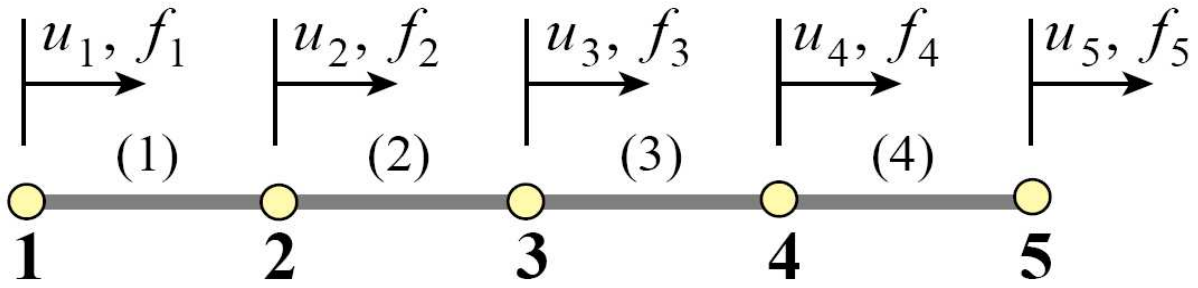
The MPE principle states that the actual displacement solution $u^*(x)$ that satisfies the governing equations is that which renders the TPE functional $\Pi[u]$ stationary:

$$\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^*$$

with respect to *admissible* variations $u = u^* + \delta u$ of the exact displacement solution $u^*(x)$

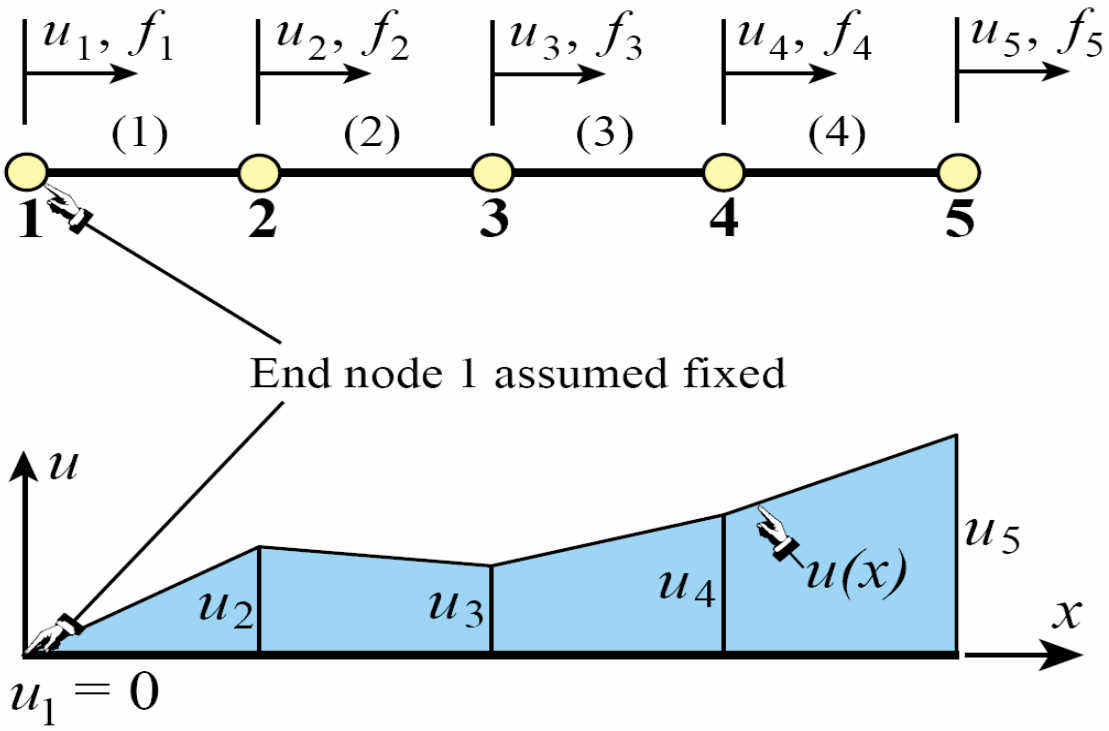
FEM: Variational Formulation of Bar Element

FEM Discretization of Bar Member



FEM: Variational Formulation of Bar Element

FEM Displacement Trial Function



Axial displacement plotted normal to x for visualization convenience

Total Potential Energy Principle and Decomposition over Elements

$$\delta\Pi = \delta U - \delta W = 0 \quad \text{iff} \quad u = u^* \quad \text{(exact solution)}$$

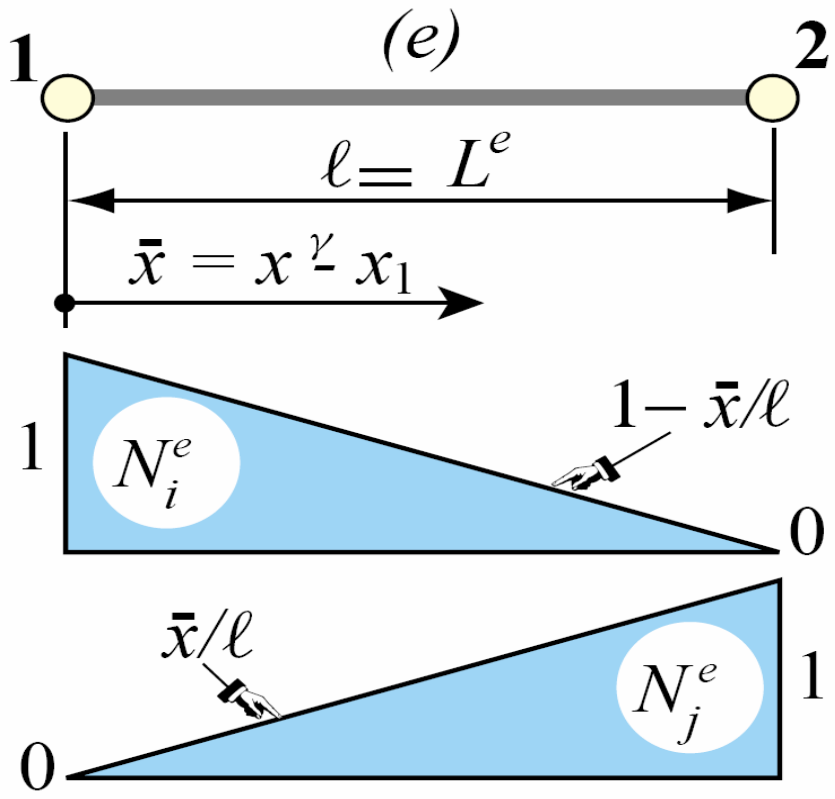
But $$\Pi = \Pi^{(1)} + \Pi^{(2)} + \ldots + \Pi^{(N^e)}$$

and $$\delta\Pi = \delta\Pi^{(1)} + \delta\Pi^{(2)} + \ldots + \delta\Pi^{(N^e)} = 0$$

From fundamental lemma of variational calculus, each ***element variation must vanish***, giving

$$\delta\Pi^e = \delta U^e - \delta W^e = 0$$

Element Shape Functions



Element Shape Functions (cont'd)

Linear displacement interpolation:

$$u^e(x) = N_1^e u_1^e + N_2^e u_2^e = [\, N_1^e \;\; N_2^e \,] \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \mathbf{N}\,\mathbf{u}^e$$

in which

$$N_1^e = 1 - \frac{x - x_1}{\ell} = 1 - \zeta, \qquad N_2^e = \frac{x - x_1}{\ell} = \zeta$$

$\zeta \equiv \frac{x - x_1}{\ell}$ dimensionless (natural) coordinate

FEM: Variational Formulation of Bar Element

Displacement Variation Process Yields the Element Stiffness Equations

$$\Pi^e = U^e - W^e \qquad \begin{cases} U^e = \frac{1}{2}(\mathbf{u}^e)^T \ \mathbf{K}^e \ \mathbf{u}^e \\ W^e = (\mathbf{u}^e)^T \ \mathbf{f}^e \end{cases}$$

$$\delta\Pi^e = 0 \ \rightarrow \ (\delta\mathbf{u}^e)^T \ [\mathbf{K}^e \mathbf{u}^e - \mathbf{f}^e] = 0$$

(Appendix D)

since $\delta\mathbf{u}^e$ is arbitrary [...] = 0

$$\mathbf{K}^e \mathbf{u}^e = \mathbf{f}^e$$

the element stiffness equations

The Bar Element Stiffness

$$U^e = \tfrac{1}{2}\int_0^\ell e\,E\,A\,e\;dx \qquad e = u'$$

$$U^e = \tfrac{1}{2}\int_0^\ell \begin{bmatrix} u_1 & u_2 \end{bmatrix} \frac{1}{\ell}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{\ell}\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} dx$$

$$U^e = \tfrac{1}{2}\begin{bmatrix} u_1^e & u_2^e \end{bmatrix} \int_0^\ell \frac{E\,A}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx \begin{bmatrix} u_1^e \\ u_2^e \end{bmatrix} = \tfrac{1}{2}\left(\mathbf{u}^e\right)^T \mathbf{K}^e\ \mathbf{u}^e$$

$$\mathbf{K}^e = \int_0^\ell E\,A\,\mathbf{B}^T\mathbf{B}\,dx = \int_0^\ell \frac{E\,A}{\ell^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx$$

If EA is constant over element

$$\mathbf{K}^e = \frac{E\,A}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The Consistent Nodal Force Vector

$$W^e = \int_0^\ell q\,u\,dx = \int_0^\ell (\mathbf{u}^e)^T\,\mathbf{N}^T q\,dx = (\mathbf{u}^e)^T \int_0^\ell q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = (\mathbf{u}^e)^T\,\mathbf{f}^e$$

Since $\mathbf{u}^e$ is arbitrary

$$\mathbf{f}^e = \int_0^\ell q \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx$$

in which $\zeta = \dfrac{x - x_1}{\ell}$

Bar Consistent Force Vector (cont'd)

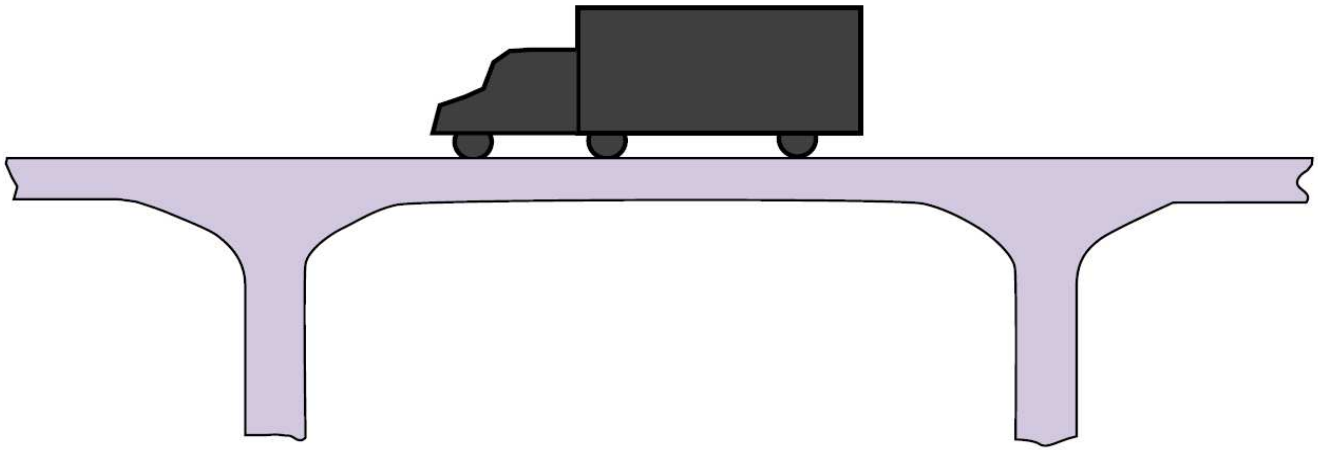
If q is ***constant*** along element

$$\mathbf{f}^e = q \int_0^\ell \begin{bmatrix} 1-\zeta \\ \zeta \end{bmatrix} dx = q\ell \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

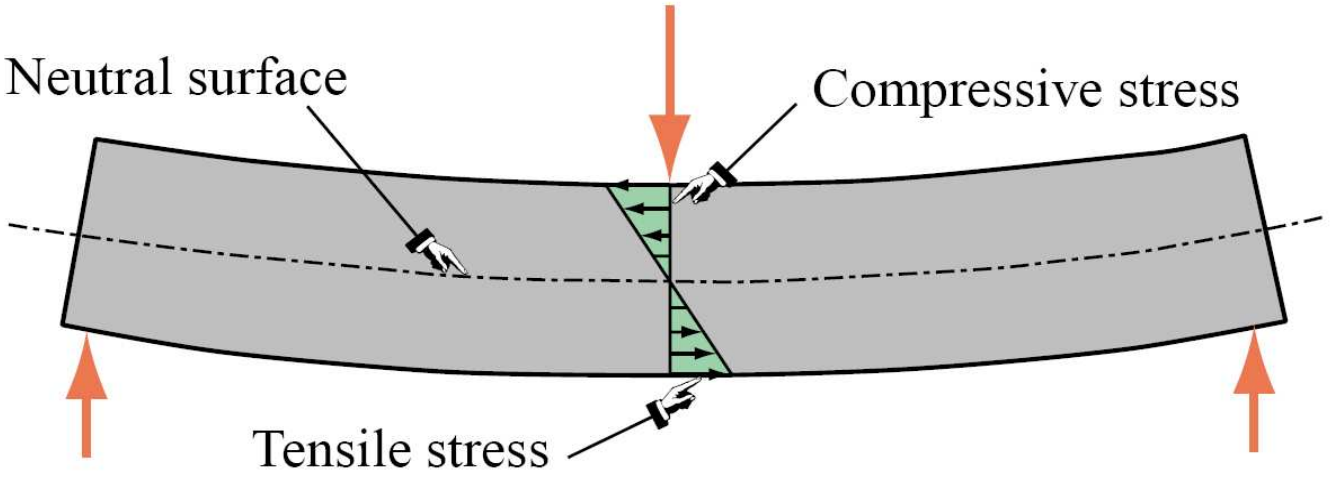
the ***same result as with EbE load lumping*** (i.e., assigning one half of the total load to each node)

FEM: Variational Formulation of Plane Beam Element

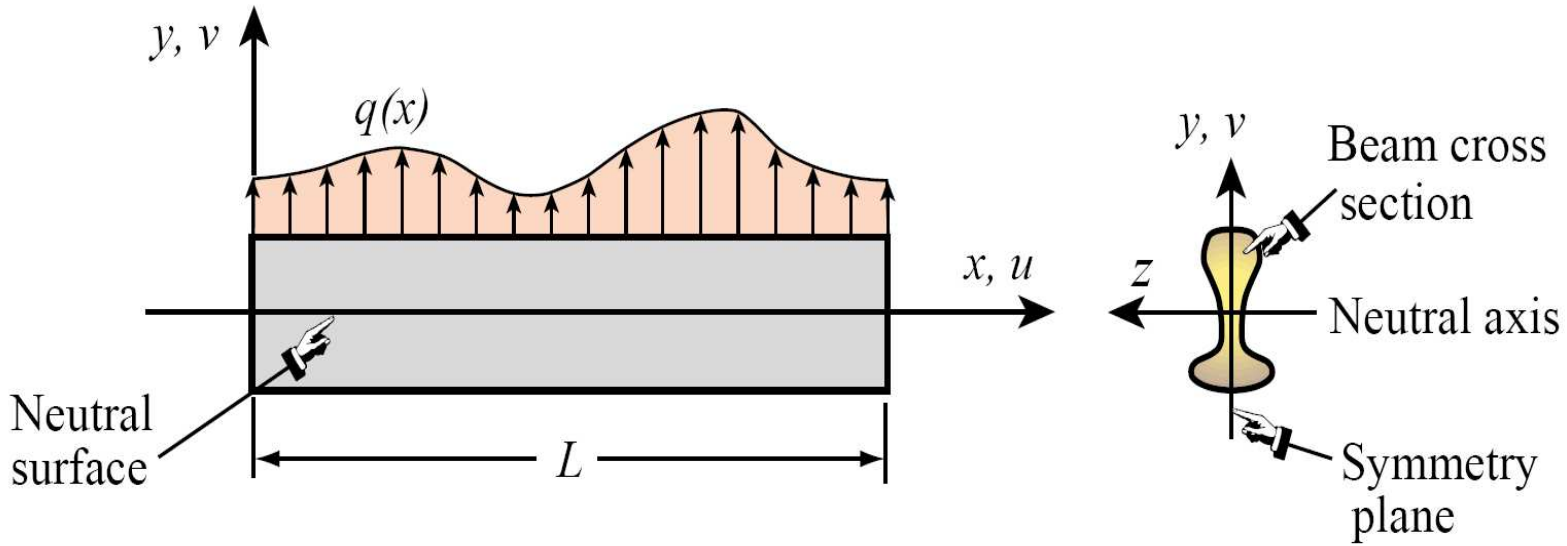
A Beam is a Structural Member Designed to Resist Primarily Transverse Loads



Transverse Loads are Transported to Supports by Flexural Action

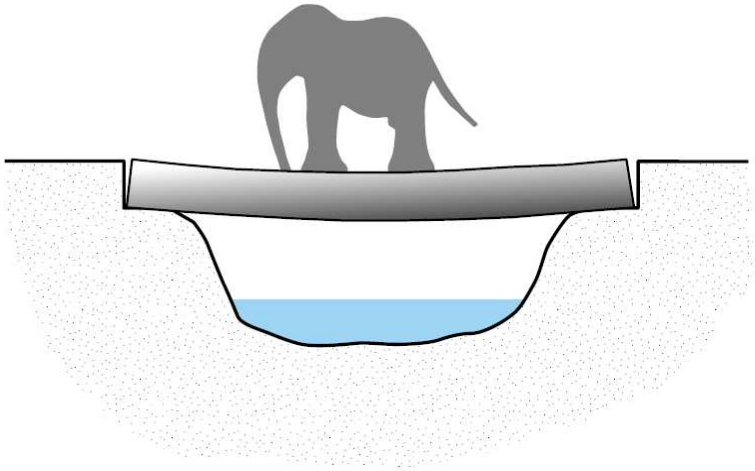


Plane Beam Terminology

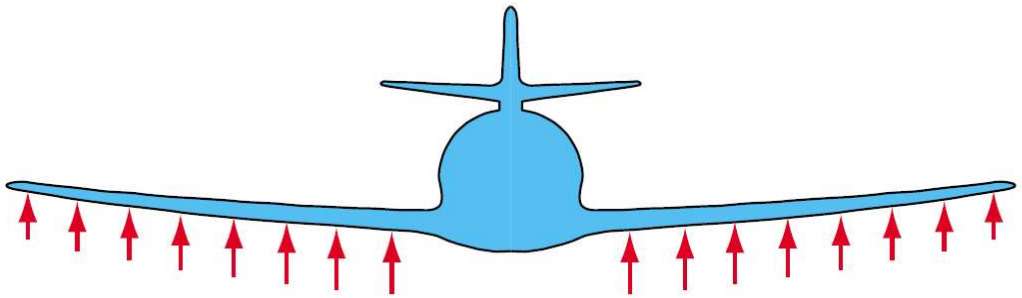


FEM: Variational Formulation of Plane Beam Element

Common Support Conditions



Simply Supported



Cantilever

Basic Relations for Bernoulli-Euler Model of Plane Beam

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} -y\dfrac{\partial v(x)}{\partial x} \\ v(x) \end{bmatrix} = \begin{bmatrix} -yv' \\ v(x) \end{bmatrix} = \begin{bmatrix} -y\theta \\ v(x) \end{bmatrix}$$

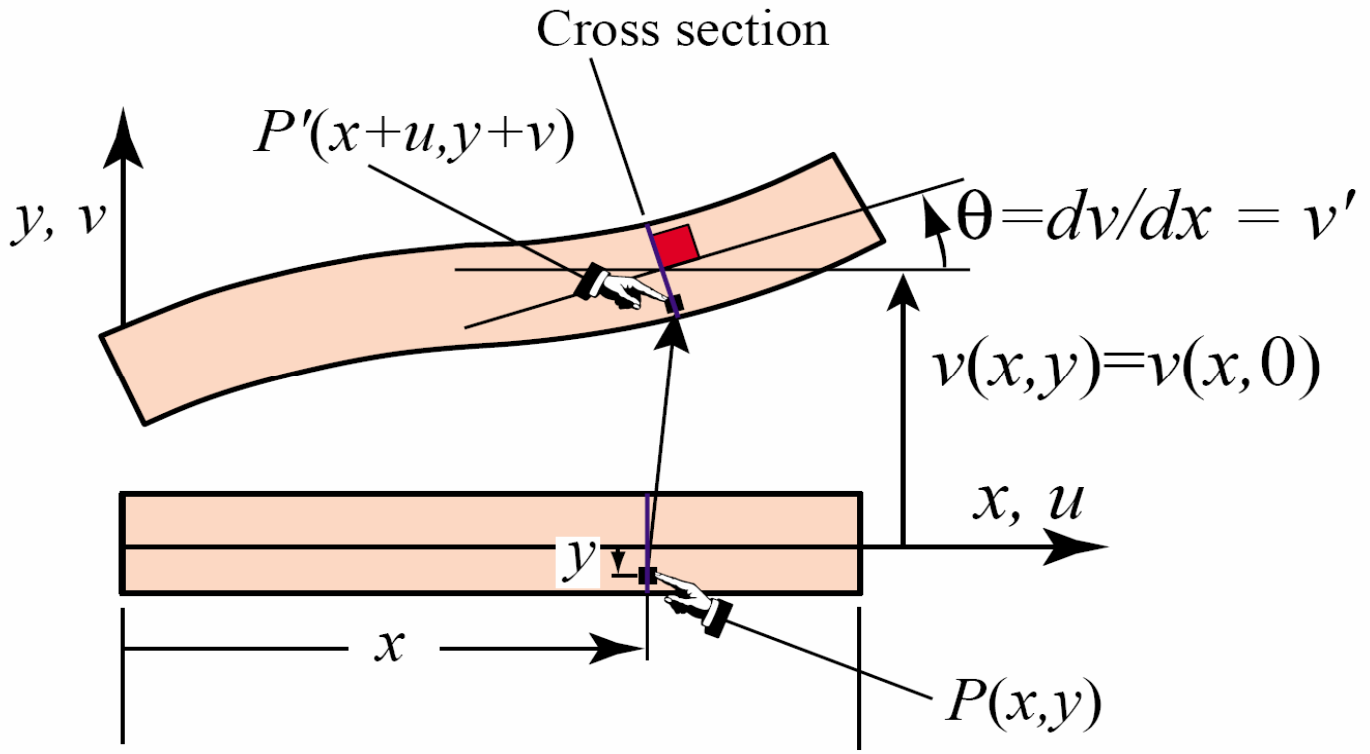
$$e = \frac{\partial u}{\partial x} = -y\frac{\partial^2 v}{\partial x^2} = -y\frac{d^2 v}{dx^2} = -y\kappa$$

$$\sigma = Ee = -Ey\frac{d^2 v}{dx^2} = -Ey\kappa$$

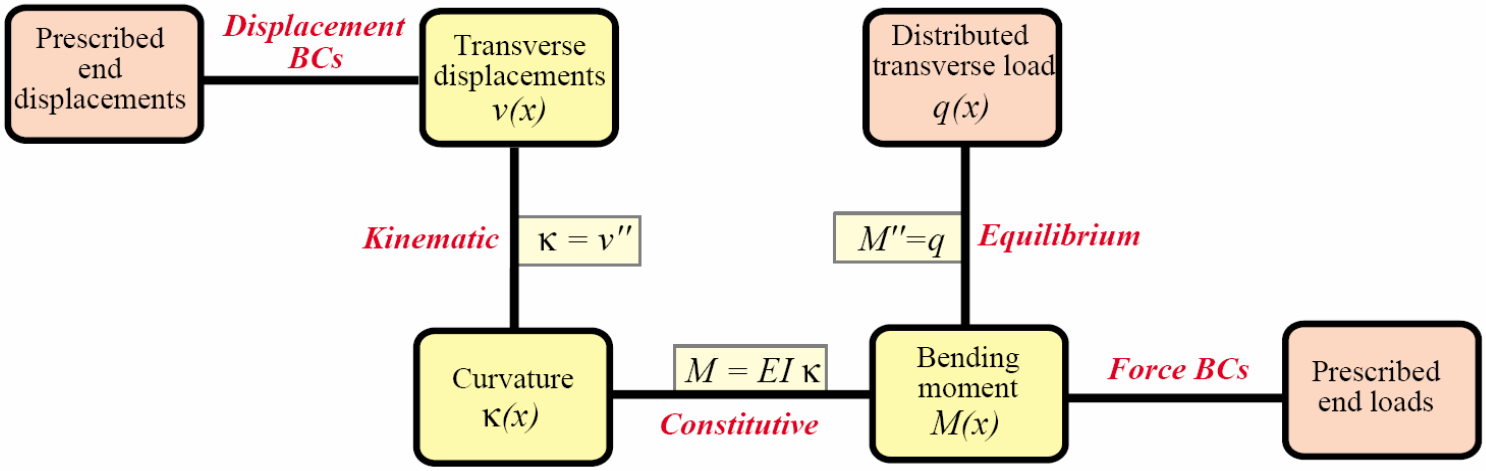
$$M = E\,I\,\kappa$$

Plus equilibrium equation $M'' = q$ (replaced by MPE principle in FEM)

Kinematics of Bernoulli-Euler Beam



Tonti Diagram for Bernoulli-Euler Model of Plane Beam (Strong Form)



Total Potential Energy of Beam Member

$$\Pi = U - W$$

Internal *External*

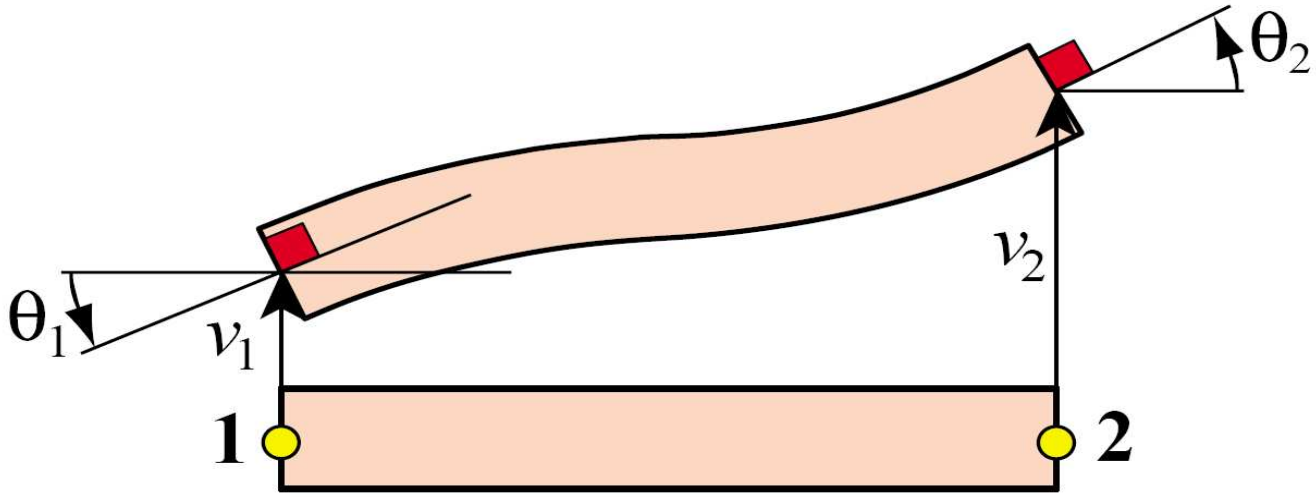
$$U = \frac{1}{2}\int_V \sigma_{xx} e_{xx}\, dV = \frac{1}{2}\int_0^L M\kappa\, dx = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2 dx$$

$$= \frac{1}{2}\int_0^L EI\kappa^2\, dx$$ *Internal energy due to bending*

$$W = \int_0^L qv\, dx$$ *External energy due to transverse load q*

FEM: Variational Formulation of Plane Beam Element

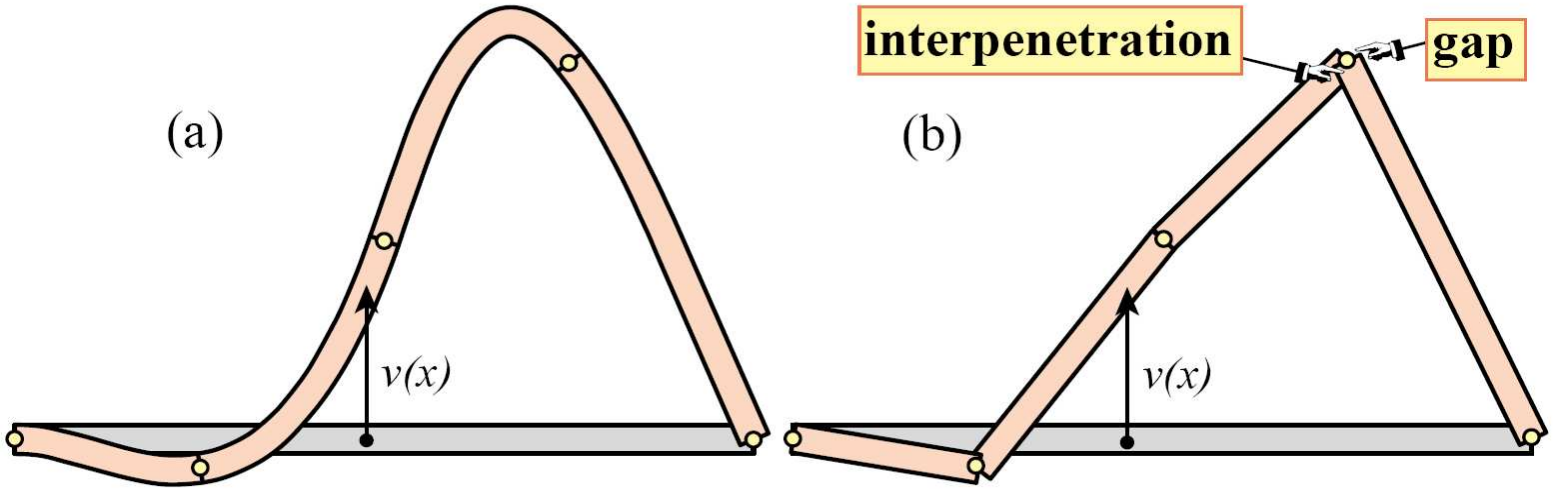
Degrees of Freedom of Plane Beam Element



$$\mathbf{u}^e = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Physical Interpretation of C^1 Continuity

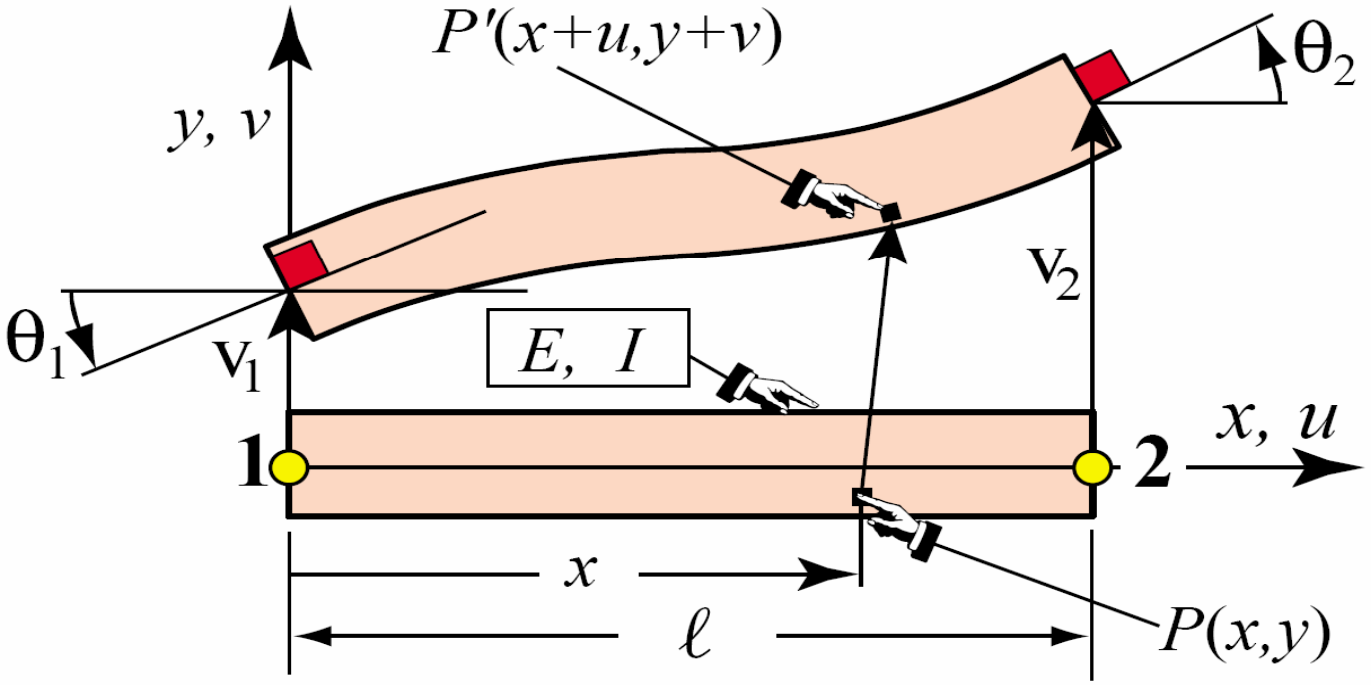
(deflections grossly exaggerated for visibility)



Piecewise cubic interpolation provides required C^1 continuity

Piecewise linear interpolation gives unacceptable C^0 continuity

Bernoulli-Euler Kinematics of Plane Beam Element



Shape Functions in Terms of Natural Coordinate ξ

$$v^e = [\, N^e_{v1} \;\; N^e_{\theta 1} \;\; N^e_{v2} \;\; N^e_{\theta 2} \,] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} = \mathbf{N} \;\; \mathbf{u}^e$$

Introduce the natural (isoparametric) coordinate

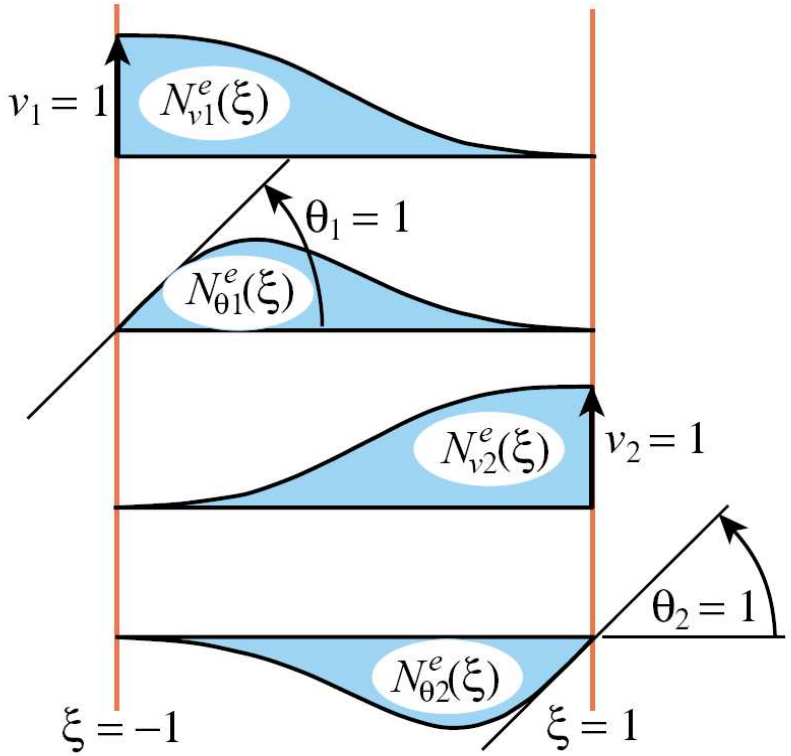
$$\xi = \frac{2x}{\ell} - 1$$

$$N^e_{v1}(\xi) = \tfrac{1}{4}(1-\xi)^2(2+\xi) \qquad N^e_{v2}(\xi) = \tfrac{1}{4}(1+\xi)^2(2-\xi)$$

$$N^e_{\theta 1}(\xi) = \tfrac{1}{8}\ell(1-\xi)^2(1+\xi) \qquad N^e_{\theta 2}(\xi) = -\tfrac{1}{8}\ell(1+\xi)^2(1-\xi)$$

Plots on next slide

Beam Element Shape Function Plots



$$N^e_{v1}(\xi) = \tfrac{1}{4}(1-\xi)^2(2+\xi)$$

$$N^e_{\theta 1}(\xi) = \tfrac{1}{8}\ell(1-\xi)^2(1+\xi)$$

$$N^e_{v2}(\xi) = \tfrac{1}{4}(1+\xi)^2(2-\xi)$$

$$N^e_{\theta 2}(\xi) = -\tfrac{1}{8}\ell(1+\xi)^2(1-\xi)$$

Getting Curvatures from the Displacement Interpolation

$$\kappa = \frac{d^2 v(x)}{dx^2} = \frac{d^2 \mathbf{N}}{dx^2}\,\mathbf{u}^e + \mathbf{N}\,\frac{d^2 \mathbf{u}^e}{dx^2}\;(\to 0) = \frac{d^2 \mathbf{N}}{dx^2}\,\mathbf{u}^e$$

$$= \left[\, \frac{d^2 N_{v1}^e}{dx^2} \quad \frac{d^2 N_{\theta 1}^e}{dx^2} \quad \frac{d^2 N_{v2}^e}{dx^2} \quad \frac{d^2 N_{\theta 2}^e}{dx^2} \,\right] \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix} \stackrel{\text{def}}{=} \mathbf{B}\,\mathbf{u}^e$$

B is the 1 x 4 curvature-displacement matrix constructed in next slide

Curvatures from Displacements (Cont'd)

To construct B, apply the chain rule

$$\frac{df(x)}{dx} = \frac{df(\xi)}{d\xi}\frac{d\xi}{dx} = \frac{2}{\ell}\frac{df(\xi)}{d\xi}$$

$$\frac{d^2f(x)}{dx^2} = \frac{d(2/\ell)}{dx}\frac{df(\xi)}{d\xi} + \frac{2}{\ell}\frac{d}{dx}\left(\frac{df(\xi)}{d\xi}\right) = \frac{4}{\ell^2}\frac{d^2f(\xi)}{d\xi}$$

($\frac{d(2/\ell)}{dx}$ equals 0)

to differentiate the shape functions. Result is

$$\mathbf{B} = \frac{1}{\ell}\left[\,6\frac{\xi}{\ell} \quad 3\xi - 1 \quad -6\frac{\xi}{\ell} \quad 3\xi + 1\,\right]$$

Element Stiffness and Consistent Node Forces

Varying the element TPE

$$\Pi^e = \tfrac{1}{2}\,\mathbf{u}^{e\,T}\,\mathbf{K}^e\,\mathbf{u}^e - \mathbf{u}^{e\,T}\,\mathbf{f}^e$$

we get

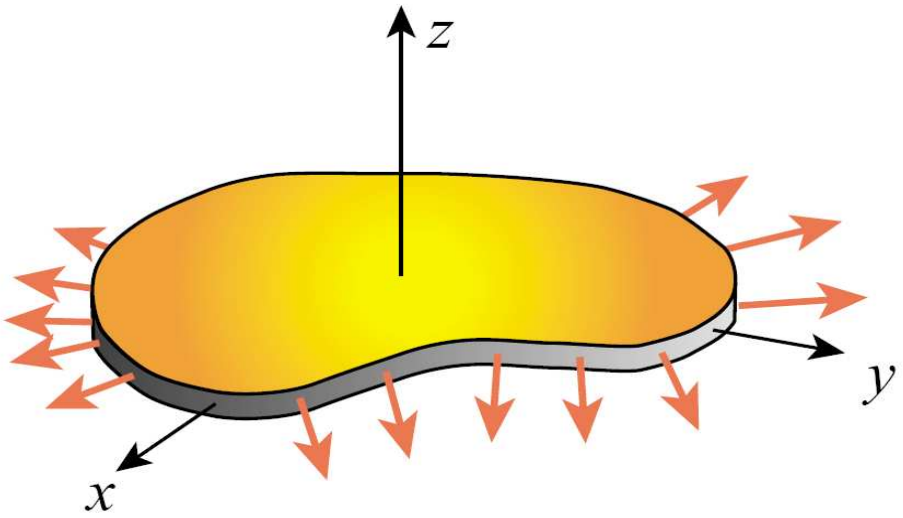
$$\mathbf{K}^e = \int_0^\ell E I\ \mathbf{B}^T\mathbf{B}\,dx = \int_{-1}^{1} E I\ \mathbf{B}^T\mathbf{B}\ \tfrac{1}{2}\ell\ d\xi$$

$$\mathbf{f}^e = \int_0^\ell \mathbf{N}^T q\,dx = \int_{-1}^{1} \mathbf{N}^T q\ \tfrac{1}{2}\ell\,d\xi$$

FEM: The Plane Stress Problem

Plate in Plane Stress

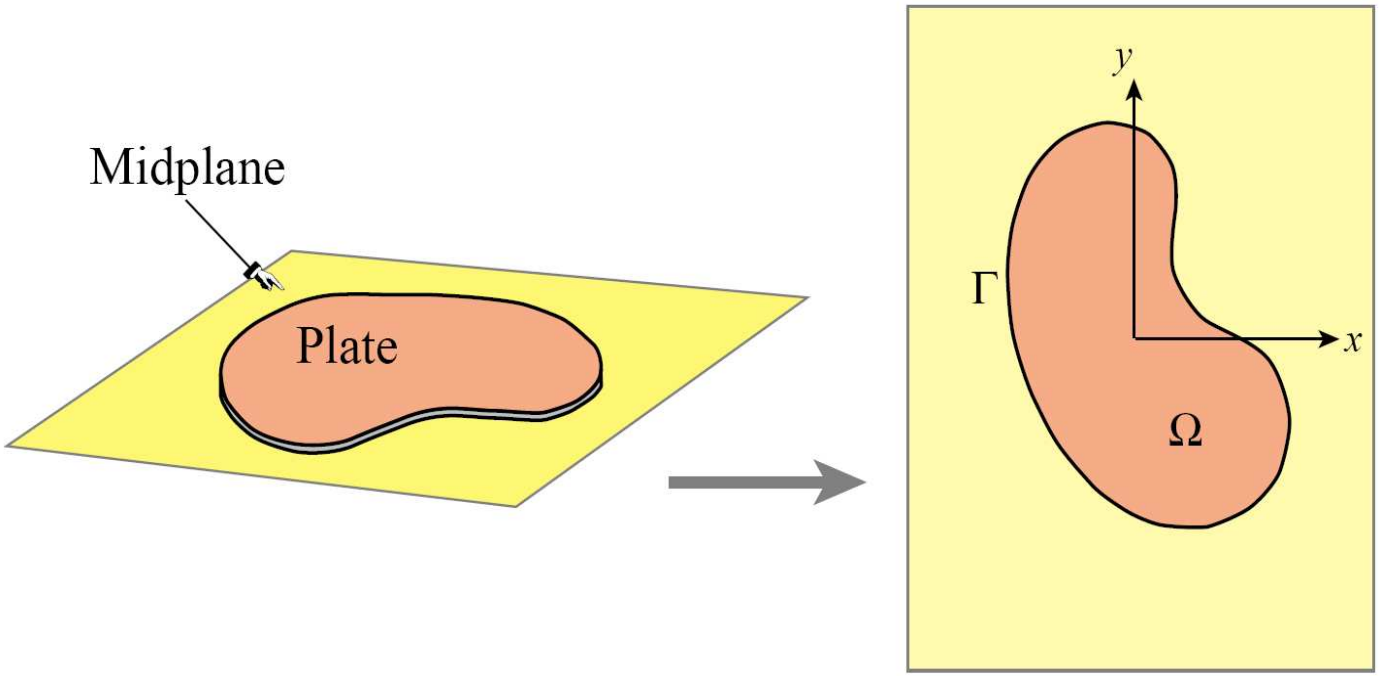
Thickness dimension
or transverse dimension



Inplane dimensions: in x,y plane

FEM: The Plane Stress Problem

Mathematical Idealization as a Two Dimensional Problem



Plane Stress Physical Assumptions

Plate is flat and has a symmetry plane (the midplane)

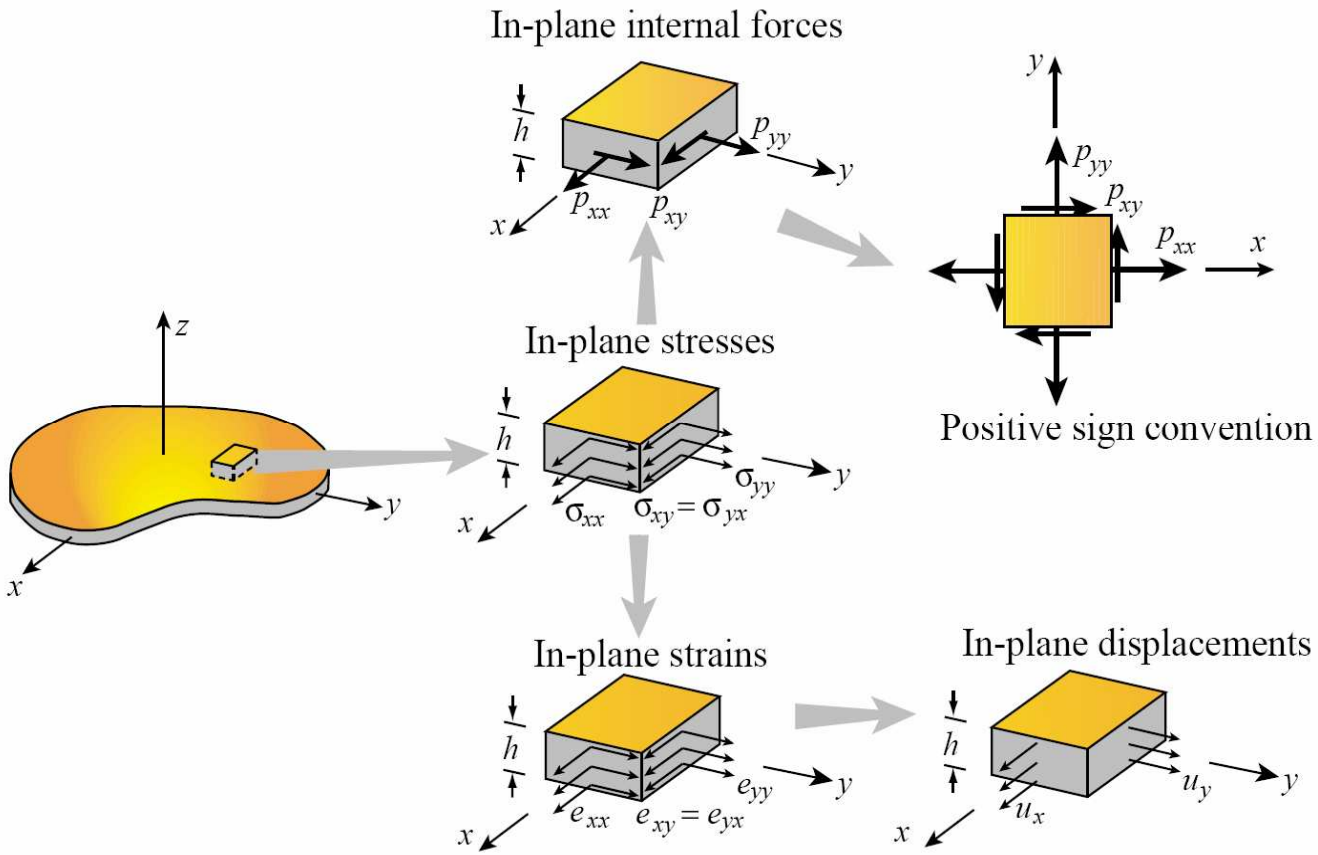
All loads and support conditions are midplane symmetric

Thickness dimension is much smaller than inplane dimensions

Inplane displacements, strains and stresses uniform through thickness

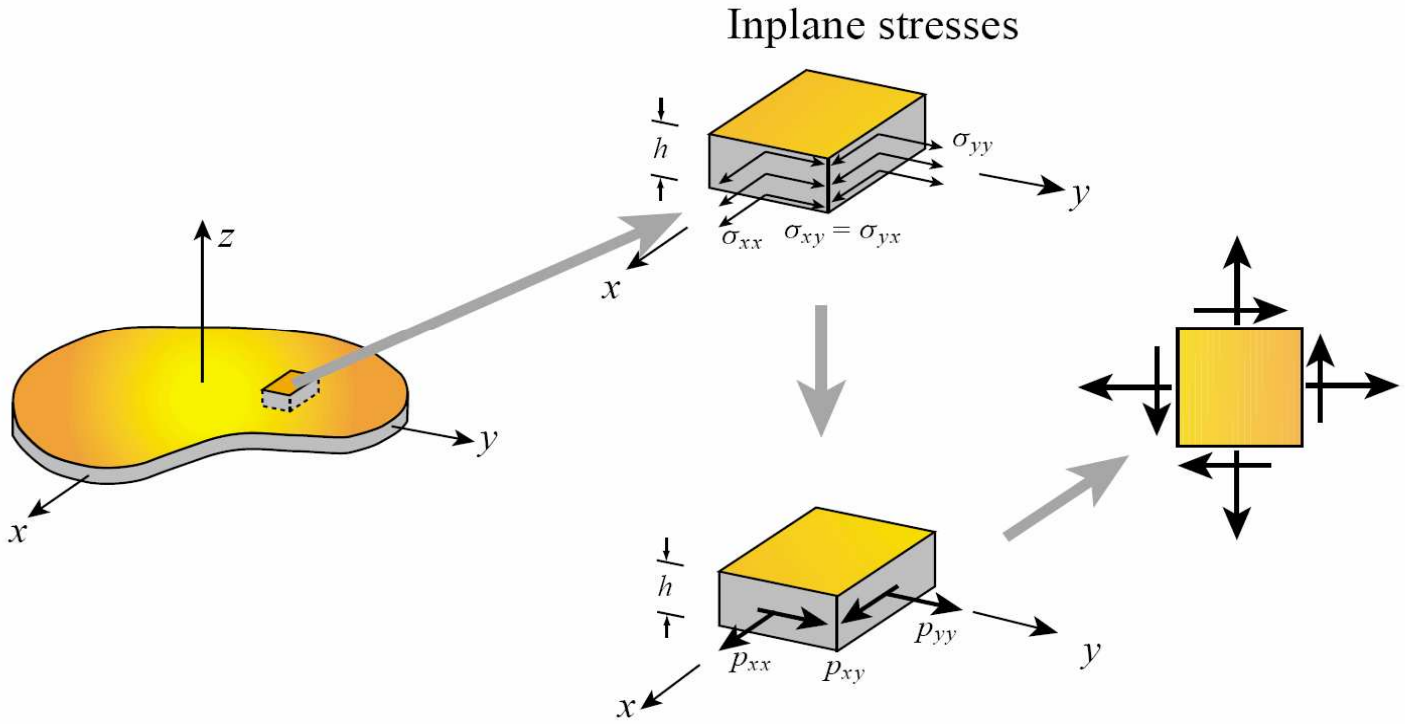
Transverse stresses σ_{zz}, σ_{xz} and σ_{yz} negligible, set to 0

Notation for Stresses, Strains, Forces, Displacements



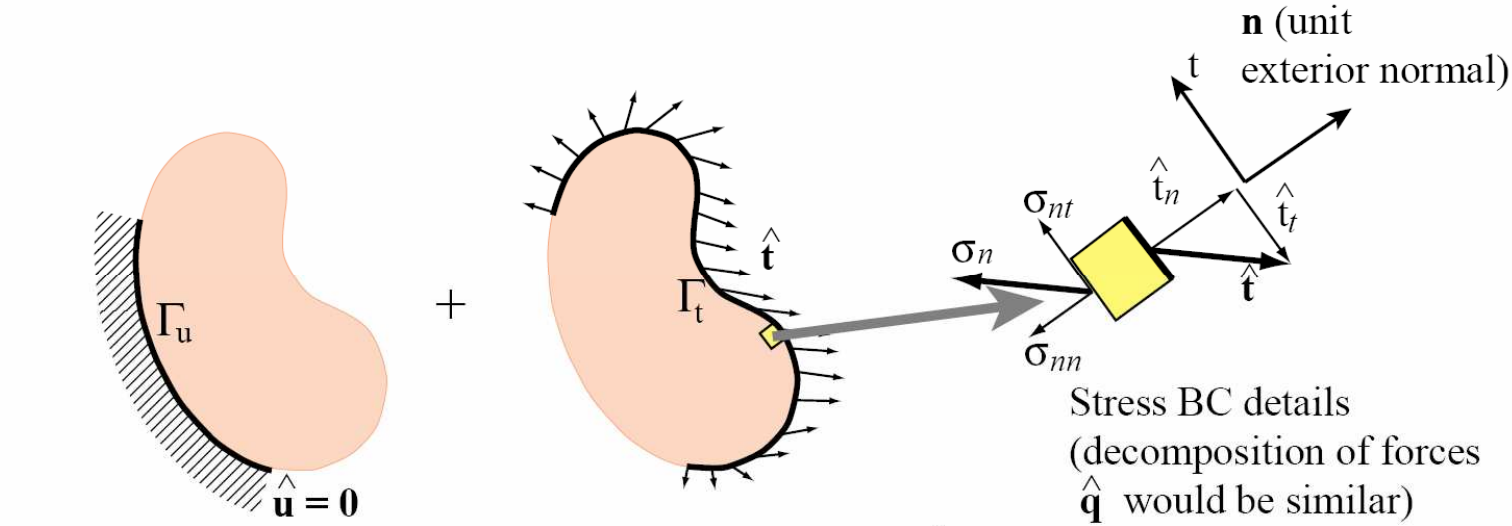
FEM: The Plane Stress Problem

Inplane Forces are Obtained by Stress Integration Through Thickness



Inplane internal forces
(also called membrane forces)

Plane Stress Boundary Conditions



Boundary displacements $\hat{\mathbf{u}}$ are prescribed on Γ_u (figure depicts fixity condition)

Boundary tractions $\hat{\mathbf{t}}$ or boundary forces $\hat{\mathbf{q}}$ are prescribed on Γ_t

The Plane Stress Problem

Given:

geometry
material properties
wall fabrication (thickness only for homogeneous plates)
applied body forces
boundary conditions:
    prescribed boundary forces or tractions
    prescribed displacements

Find:

inplane displacements
inplane strains
inplane stresses and/or internal forces

Matrix Notation for Internal Fields

$$\mathbf{u}(x, y) = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix}$$

displacements

$$\mathbf{e}(x, y) = \begin{bmatrix} e_{xx}(x, y) \\ e_{yy}(x, y) \\ 2e_{xy}(x, y) \end{bmatrix}$$

strains (factor of 2 in e_{xy} simplifies "energy dot products")

$$\boldsymbol{\sigma}(x, y) = \begin{bmatrix} \sigma_{xx}(x, y) \\ \sigma_{yy}(x, y) \\ \sigma_{xy}(x, y) \end{bmatrix}$$

stresses

FEM: The Plane Stress Problem

Governing Plane Stress Elasticity Equations in Matrix Form

$$\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

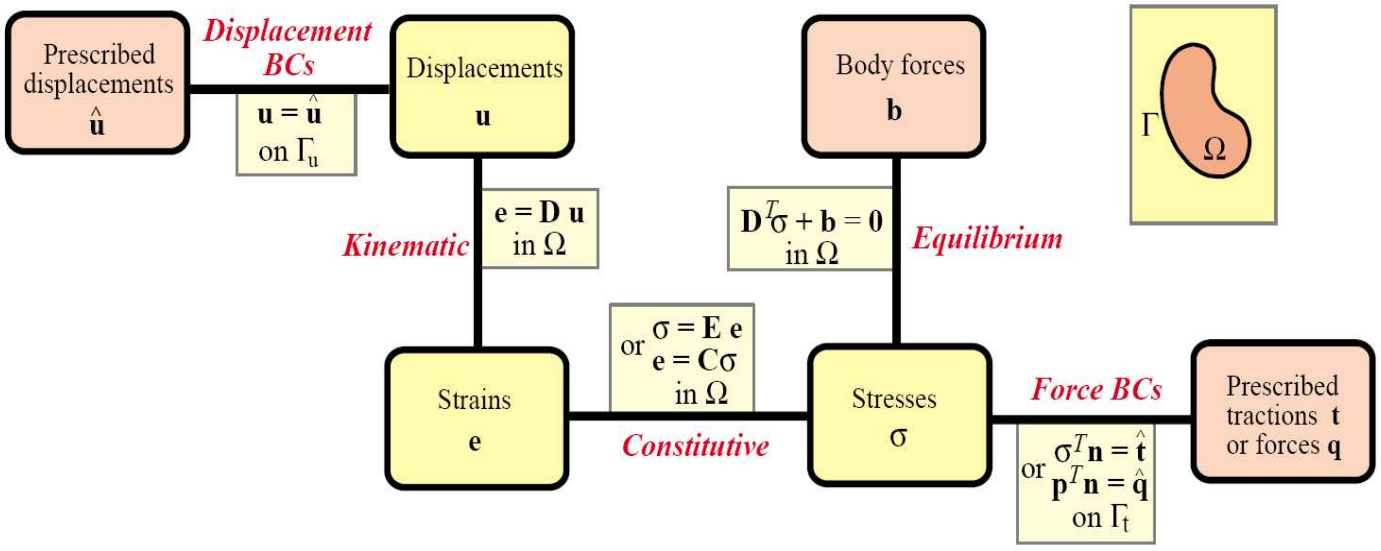
$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

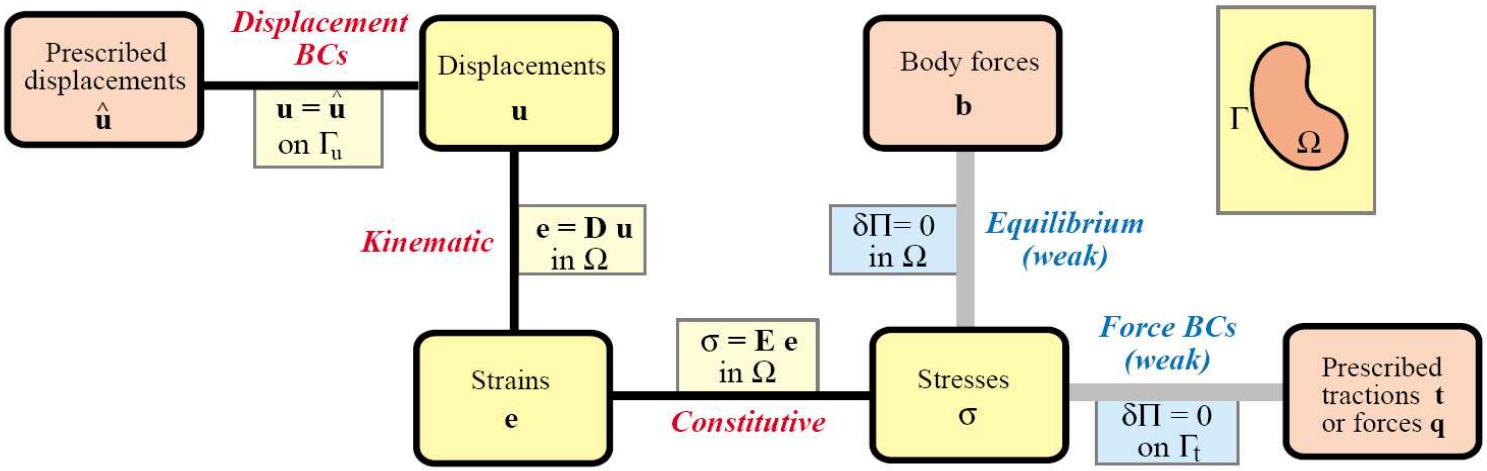
$$\mathbf{e} = \mathbf{D}\mathbf{u} \qquad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} \qquad \mathbf{D}^T\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$$

Strong-Form Tonti Diagram of Plane Stress Governing Equations



FEM: The Plane Stress Problem

TPE-Based Weak Form Diagram of Plane Stress Governing Equations



FEM: The Plane Stress Problem

Total Potential Energy of Plate in Plane Stress

$$\Pi = U - W$$

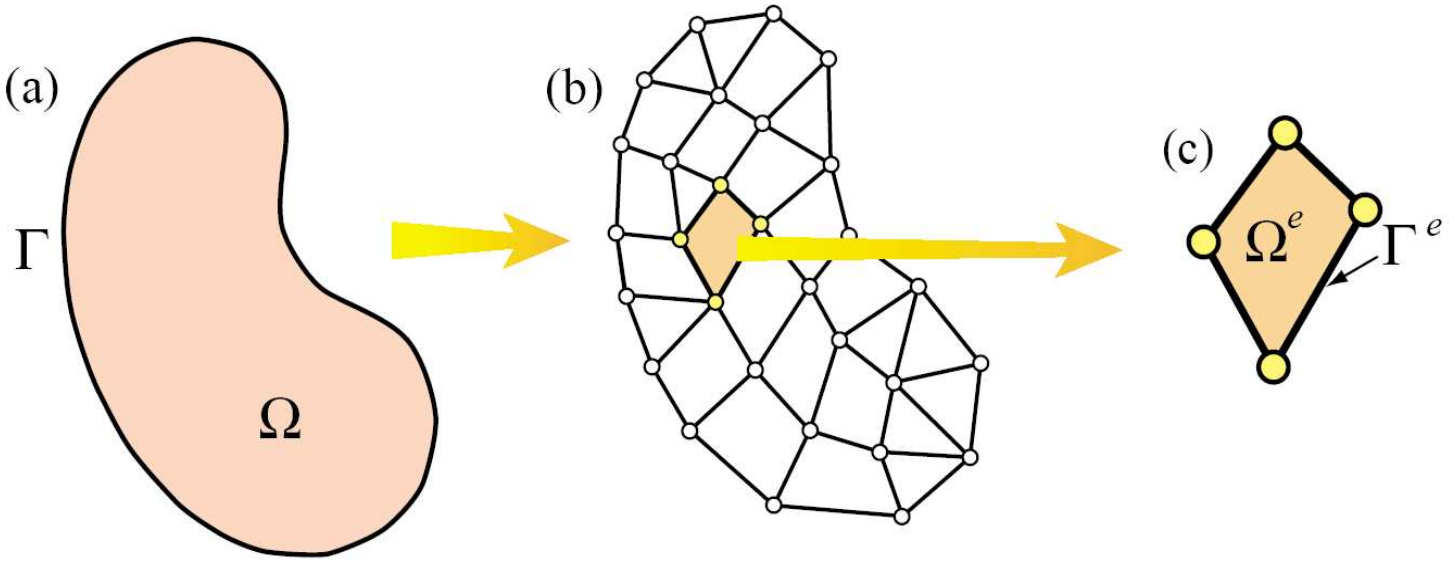
$$U = \frac{1}{2}\int_{\Omega} h\,\boldsymbol{\sigma}^T \mathbf{e}\, d\Omega = \frac{1}{2}\int_{\Omega} h\,\mathbf{e}^T \mathbf{E}\mathbf{e}\, d\Omega$$

$$W = \int_{\Omega} h\,\mathbf{u}^T \mathbf{b}\, d\Omega + \int_{\Gamma_t} h\,\mathbf{u}^T \hat{\mathbf{t}}\, d\Gamma$$

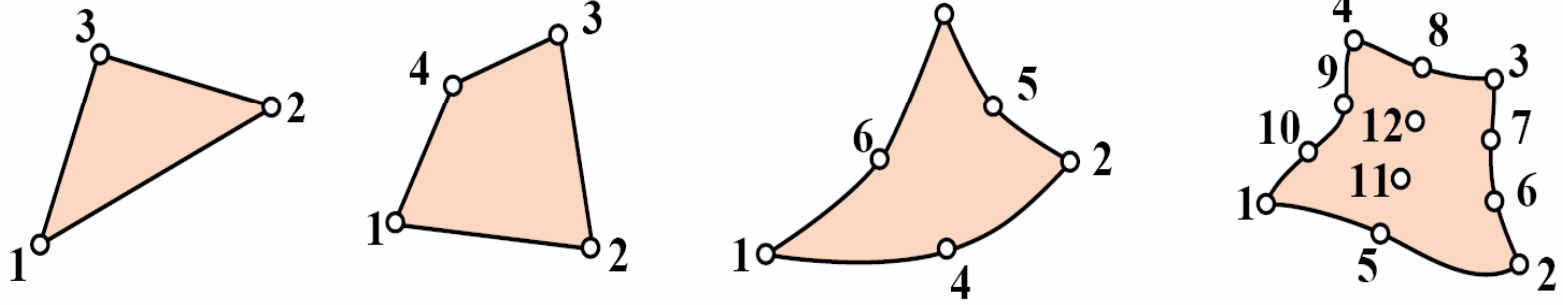
body forces *boundary tractions*

FEM: The Plane Stress Problem

Discretization into Plane Stress Finite Elements



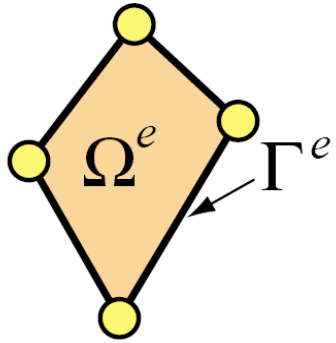
Plane Stress Element Geometries and Node Configurations



$n = 3$ $n = 4$ $n = 6$ $n = 12$

FEM: The Plane Stress Problem

Total Potential Energy of Plane Stress Element

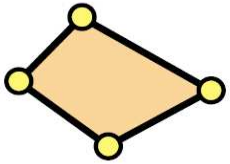


$$\Pi^{e\,\gamma} = \quad U^e \quad - W^e$$

$$U^e \quad = \frac{1}{2}\int_{\Omega^e} h\,\boldsymbol{\sigma}^T\mathbf{e}\,d\Omega^e = \frac{1}{2}\int_{\Omega^e} h\,\mathbf{e}^T\mathbf{E}\,\mathbf{e}\,d\Omega^e$$

$$W^e = \quad \int_{\Omega^e} h\,\mathbf{u}^T\mathbf{b}\,d\Omega^e \quad + \int_{\Gamma^e} h\,\mathbf{u}^T\mathbf{t}\,d\Gamma^e$$

Constructing a Displacement Assumed Element



n nodes, *n*=4 in figure

Node displacement vector

$$\mathbf{u}^e = [\, u_{x1} \quad u_{y1} \quad u_{x2} \quad \ldots \quad u_{xn} \quad u_{yn} \,]^T$$

Displacement interpolation over element

$$\mathbf{u}(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix} = \begin{bmatrix} N_1^e & 0 & N_2^e & 0 & \ldots & N_n^e & 0 \\ 0 & N_1^e & 0 & N_2^e & \ldots & 0 & N_n^e \end{bmatrix} \mathbf{u}^e$$

$$= \mathbf{N}\ \mathbf{u}^e$$

N is called the shape function matrix
It has order 2 x 2*n*

Element Construction (cont'd)

Differentiate the displacement interpolation wrt x,y to get the strain-displacement relation

$$\mathbf{e}(x,y) = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \cdots & \frac{\partial N_n^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & \cdots & 0 & \frac{\partial N_n^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & \cdots & \frac{\partial N_n^e}{\partial y} & \frac{\partial N_n^e}{\partial x} \end{bmatrix} \mathbf{u}^e = \mathbf{B}\,\mathbf{u}^e$$

B is called the strain-displacement matrix
It has order 3 x 2*n*

Element Construction (cont'd)

Element total potential energy

$$\Pi^e = \tfrac{1}{2}\,\mathbf{u}^{eT}\mathbf{K}^e\,\mathbf{u}^e - \mathbf{u}^{eT}\mathbf{f}^e$$

Element stiffness matrix

$$\mathbf{K}^e = \int_{\Omega^e} h\,\mathbf{B}^T\mathbf{E}\,\mathbf{B}\,d\Omega^e$$

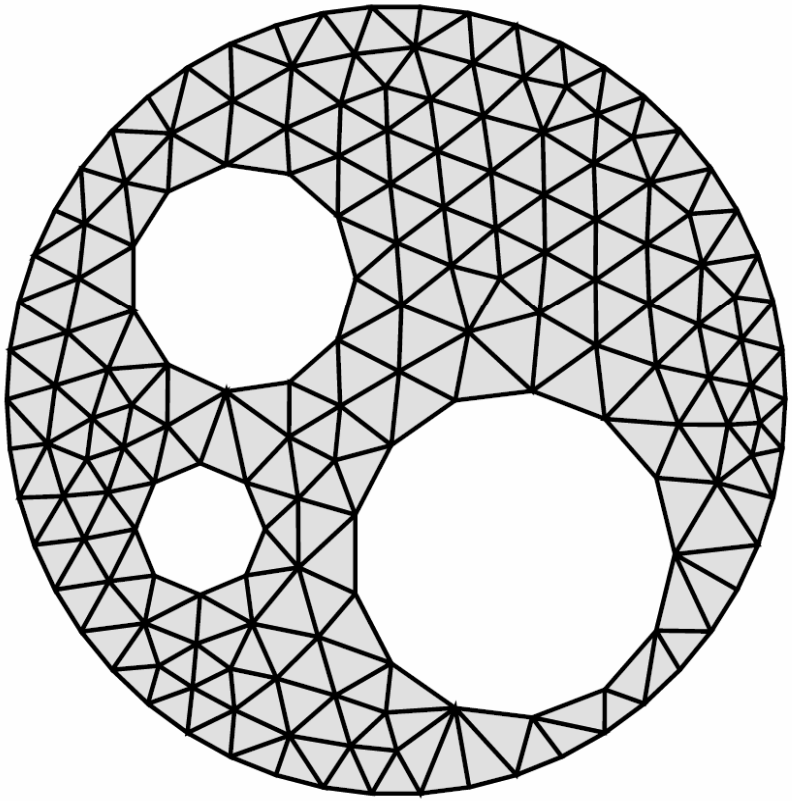
Consistent node force vector

$$\mathbf{f}^e = \int_{\Omega^e} h\,\mathbf{N}^T\mathbf{b}\,d\Omega^e + \int_{\Gamma^e} h\,\mathbf{N}^T\hat{\mathbf{t}}\,d\Gamma^e$$

due to: body force due to: surface tractions

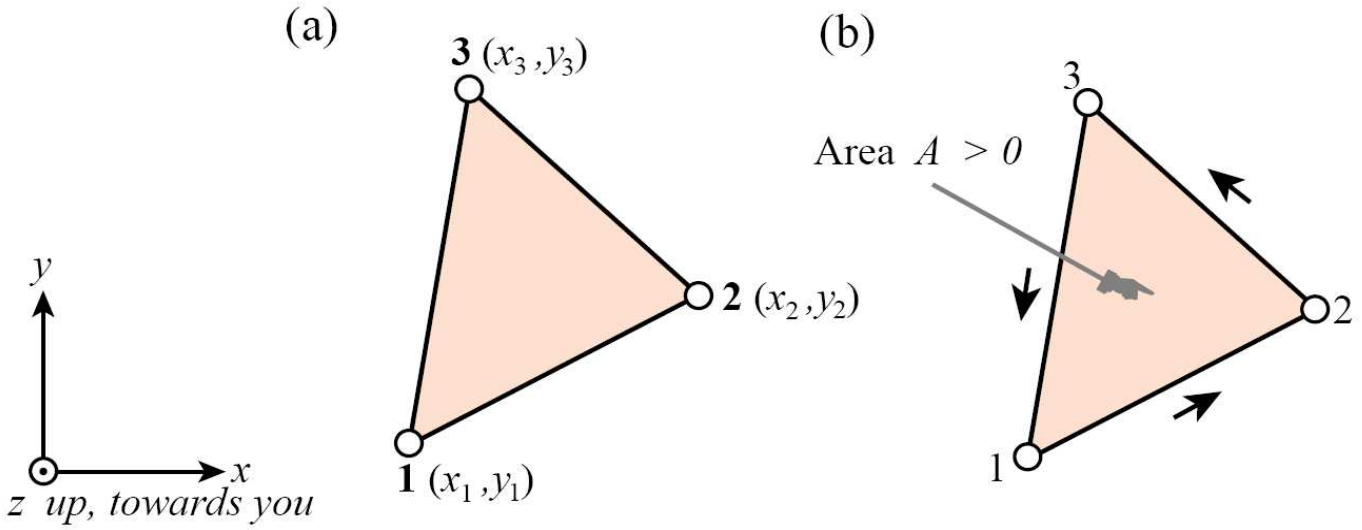
FEM: Three Node Plane Stress Triangles

Triangles are Still Popular Because of Geometric Versatility and Ease of Automated Mesh Generation



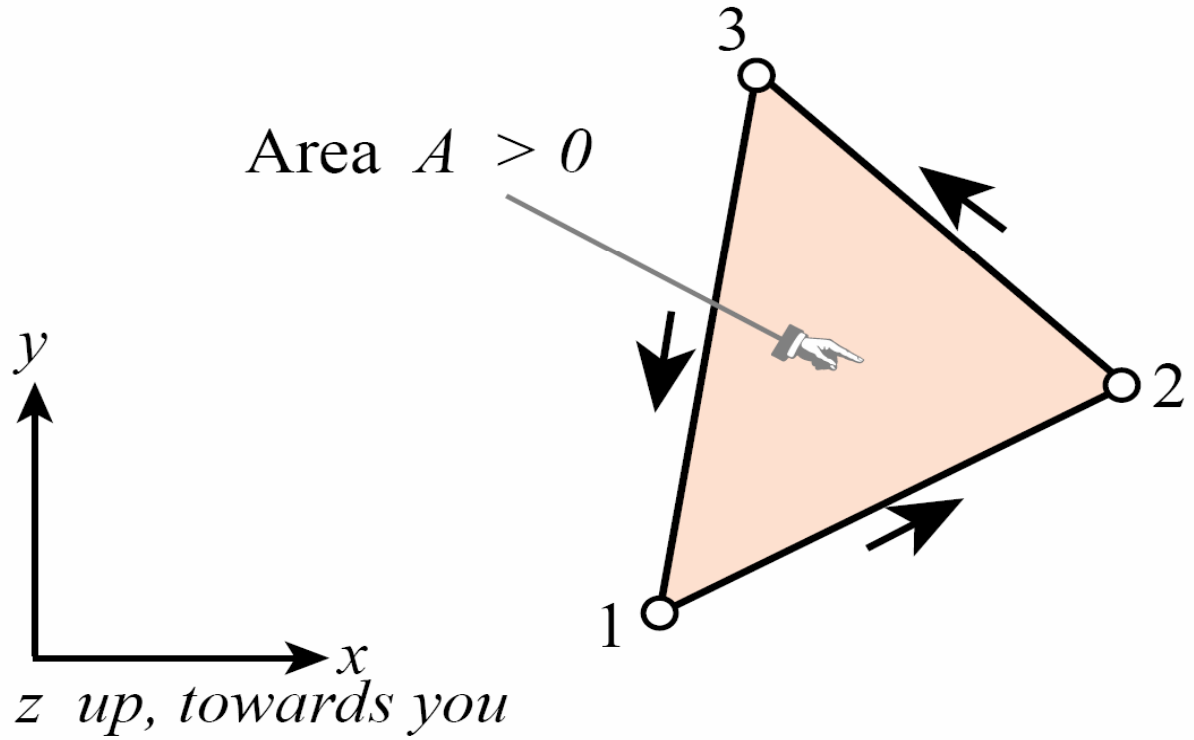
FEM: Three Node Plane Stress Triangles

The Turner Triangle Geometry and its Nodal Configuration



FEM: Three Node Plane Stress Triangles

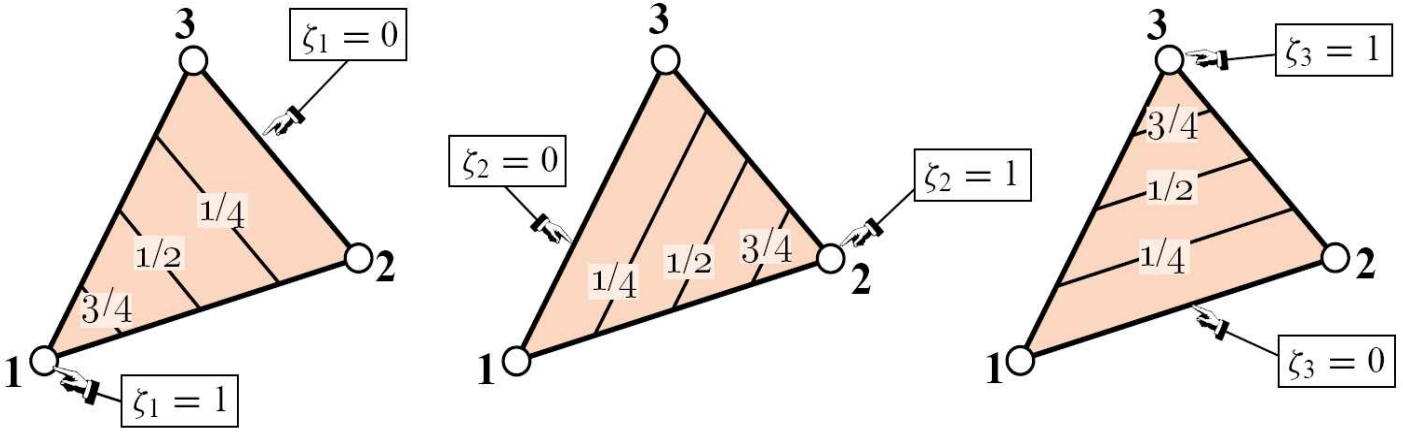
Positive Element Boundary Traversal Convention



FEM: Three Node Plane Stress Triangles

Triangular Coordinates

ζ_1 ζ_2 ζ_3



$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

Using Triangular Coordinates to Formulate Linear Interpolation

Cartesian form

$$f(x, y) = a_0 + a_1 x + a_2 y$$

Variation defined by 3 corner values $f_1,\ f_2,\ f_3$

Natural form

$$f(\zeta_1, \zeta_2, \zeta_3) = f_1\zeta_1 + f_2\zeta_2 + f_3\zeta_3 = [\, f_1 \quad f_2 \quad f_3 \,] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

$$= [\, \zeta_1 \quad \zeta_2 \quad \zeta_3 \,] \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Relating Triangular & Cartesian Coordinates I

Triangular to Cartesian:

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Relating Triangular & Cartesian Coordinates II

Cartesian to Triangular:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} 2A_{23} & y_{23} & x_{32} \\ 2A_{31} & y_{31} & x_{13} \\ 2A_{12} & y_{12} & x_{21} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

Here $x_{jk} = x_j - x_k \qquad y_{jk} = y_j - y_k$

A_{jk} denotes the area subtended by corners j, k and the origin of the x-y system

Partial Derivatives for Cartesian-Triangular Coordinate Transformations

$$\frac{\partial x}{\partial \zeta_i} = x_i \qquad \frac{\partial y}{\partial \zeta_i} = y_i$$

$$2A\frac{\partial \zeta_i}{\partial x} = y_{jk} \qquad 2A\frac{\partial \zeta_i}{\partial y} = x_{kj}$$

$$i = 1,2,3 \qquad j = 2,3,1 \qquad k = 3,1,2$$

FEM: Three Node Plane Stress Triangles

Cartesian Partial Derivatives of a General Triangular-Coordinates Function

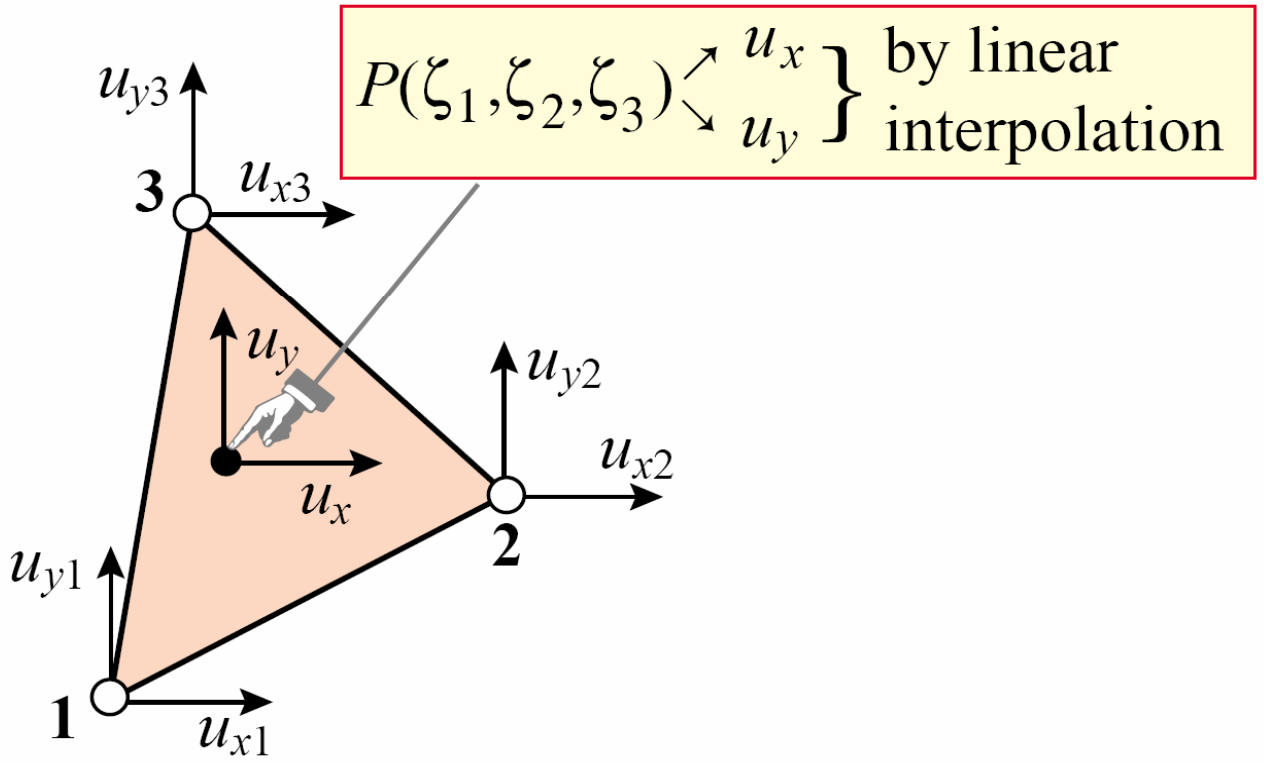
$$f = f(\zeta_1, \zeta_2, \zeta_3)$$

$$\frac{\partial f}{\partial x} = \frac{1}{2A}\left(\frac{\partial f}{\partial \zeta_1} y_{23} + \frac{\partial f}{\partial \zeta_2} y_{31} + \frac{\partial f}{\partial \zeta_3} y_{12}\right)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2A}\left(\frac{\partial f}{\partial \zeta_1} x_{32} + \frac{\partial f}{\partial \zeta_2} x_{13} + \frac{\partial f}{\partial \zeta_3} x_{21}\right)$$

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial \zeta_1} \\ \frac{\partial f}{\partial \zeta_2} \\ \frac{\partial f}{\partial \zeta_3} \end{bmatrix}$$

Displacement Interpolation over Turner Triangle



FEM: Three Node Plane Stress Triangles

Displacement Interpolation (Cont'd)

$$u_x = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$
$$u_y = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

In matrix form

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{N}\ \mathbf{u}^e$$

The shape functions are $N_i = \zeta_i \ , \quad i = 1,2,3$

FEM: Three Node Plane Stress Triangles

Strain-Displacement Relations

$$\mathbf{e} = \mathbf{D}\,\mathbf{N}\,\mathbf{u}^e = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \end{bmatrix} = \mathbf{B}\,\mathbf{u}^e$$

Stress-Strain Relations

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}\begin{bmatrix} e_{xx} \\ e_{yy} \\ 2e_{xy} \end{bmatrix} = \mathbf{E}\mathbf{e}$$

FEM: Three Node Plane Stress Triangles

Element Stiffness Matrix of Turner Triangle

$$\mathbf{K}^e = \int_{\Omega^e} h\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,d\Omega$$

Because **B** and **E** are constant over the triangle area:

$$\mathbf{K}^e = \mathbf{B}^T\mathbf{E}\mathbf{B}\int_{\Omega^e} h\,d\Omega$$

$$= \frac{1}{4A^2}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \int_{\Omega^e} h\,d\Omega$$

FEM: Three Node Plane Stress Triangles

If the Plate Thickness h is Constant

$$\mathbf{K}^e = \frac{h}{4A}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

FEM: Three Node Plane Stress Triangles

Consistent Node Force Vector for Body Forces

$$\mathbf{b} = \begin{bmatrix} b_x \\ b_y \end{bmatrix}$$

$$\mathbf{f}^e = \int_{\Omega^e} h\,(\mathbf{N}^e)^T\,\mathbf{b}\,d\Omega = \int_{\Omega^e} h \begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_1 \\ \zeta_2 & 0 \\ 0 & \zeta_2 \\ \zeta_3 & 0 \\ 0 & \zeta_3 \end{bmatrix} \mathbf{b}\,d\Omega$$

FEM: Three Node Plane Stress Triangles

If Body Forces are Constant over Element

Using *

$$\int_{\Omega^e} \zeta_1 \, d\Omega = \int_{\Omega^e} \zeta_2 \, d\Omega = \int_{\Omega^e} \zeta_3 \, d\Omega = \tfrac{1}{3} A$$

we get

$$\mathbf{f}^e = \frac{Ah}{3} \begin{bmatrix} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{bmatrix}$$

same result as straightforward "load lumping"

* These area integrals are instances of a general formula given on the next slide

FEM: Three Node Plane Stress Triangles

Integrating Triangular Coordinate Monomials

Integrals

$$\int_{\Omega^e} \zeta_1 \, d\Omega \;=\; \int_{\Omega^e} \zeta_2 \, d\Omega \;=\; \int_{\Omega^e} \zeta_3 \, d\Omega \;=\; \tfrac{1}{3} A \qquad (*)$$

used in the previous slide are instances of the general formula

$$\frac{1}{2A} \int_{\Omega^e} \zeta_1^i \, \zeta_2^j \, \zeta_3^k \, d\Omega \;=\; \frac{i!\, j!\, k!}{(i + j + k + 2)!}$$

where exponents i, j, k are nonnegative integers: $i \geq 0, j \geq 0, k \geq 0$.
To get (*) set $i = 1, j = k = 0$, then permute cyclically.

The general formula is valid for triangles with *straight* sides and *constant* metric. It does not extend to curved-side triangles, a restriction not mentioned in FEM textbooks.

Isoparametric Representation of Finite Elements

Element geometry and displacements are represented by same set of shape functions (iso = same)

FEM: Isoparametric Representation

Strong and Weak Points of Isoparametric Formulations for Structural Mechanics

Advantages

Unification: same steps for all iso-P elements
No need to distinguish straight vs. curved side elements
Quick construction of shape functions

Disadvantages

Low-order iso-P elements may be poor performers (overstiff)
Method does not extend to problems with variational index higher than 1 (e.g., plate bending and shells)

FEM: Isoparametric Representation

Before Isoparametric Concept was Discovered, FEM Developers Did "SuperParametric" Elems

Element shape functions refined, more nodes and DOFs added

But element geometry was kept simple with straight sides

FEM: Isoparametric Representation

For the 3 Node Linear Triangle, Recall:

Geometric Description

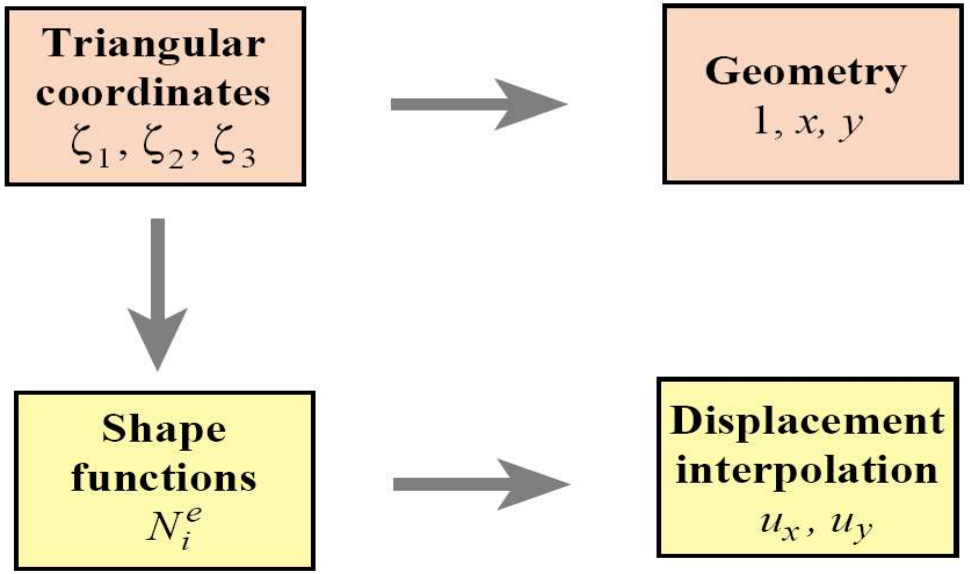
$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

Displacement Interpolation

$$u_x = u_{x1} N_1^e + u_{x2} N_2^e + u_{x3} N_3^e = u_{x1}\zeta_1 + u_{x2}\zeta_2 + u_{x3}\zeta_3$$
$$u_y = u_{y1} N_1^e + u_{y2} N_2^e + u_{y3} N_3^e = u_{y1}\zeta_1 + u_{y2}\zeta_2 + u_{y3}\zeta_3$$

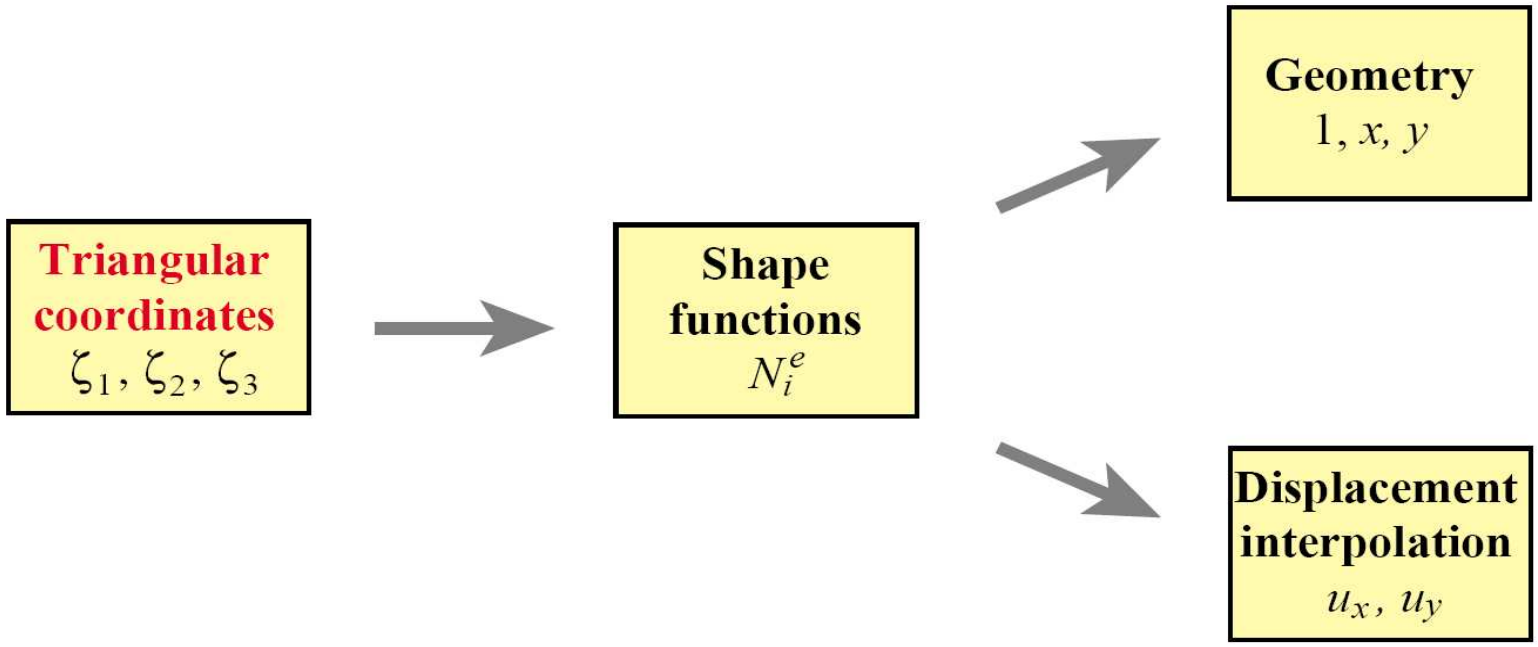
FEM: Isoparametric Representation

SuperParametric Representation (Triangles)



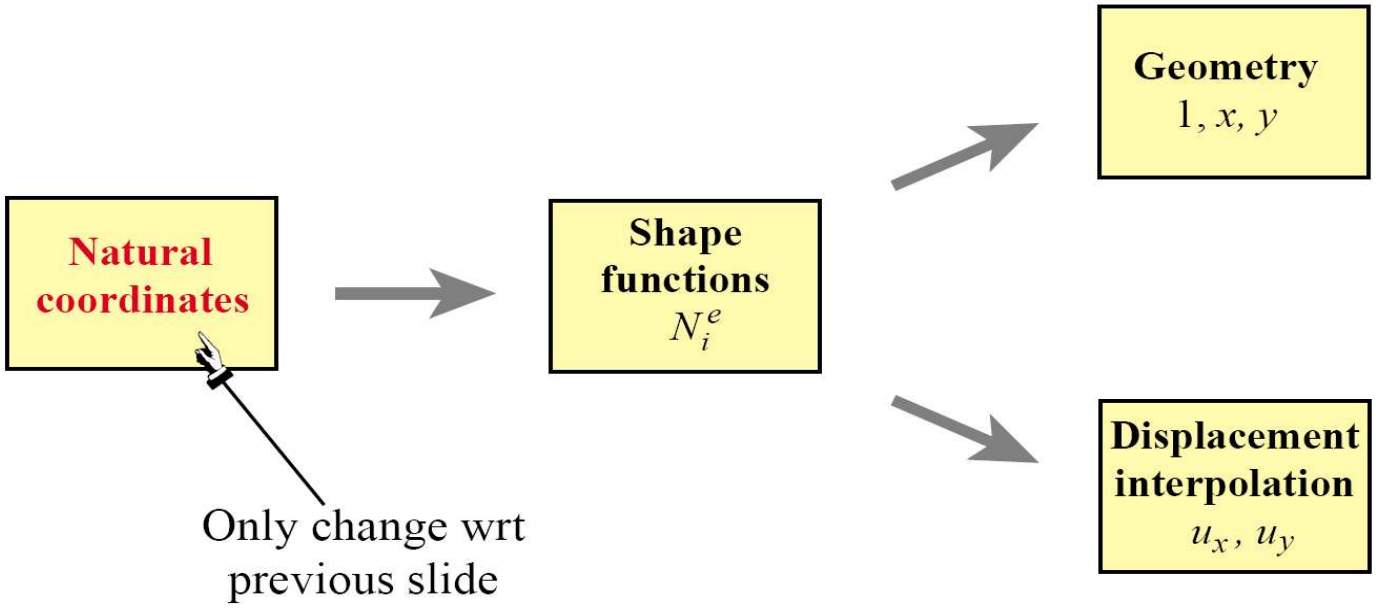
FEM: Isoparametric Representation

Isoparametric Representation (Iso=Equal) for Triangular Elements



FEM: Isoparametric Representation

Isoparametric Representation for *any* 2D Element



In 3D: $1,x,y$ **becomes** $1,x,y,z$ **etc**

FEM: Isoparametric Representation

Iso-P Representation of 2D Plane Stress Elements with n Nodes

Element Geometry:

$$1 = \sum_{i=1}^{n} N_i^e \ , \qquad x = \sum_{i=1}^{n} x_i N_i^e \ , \qquad y = \sum_{i=1}^{n} y_i N_i^e$$

Displacement Interpolation

$$u_x = \sum_{i=1}^{n} u_{xi} N_i^e \ , \qquad u_y = \sum_{i=1}^{n} u_{yi} N_i^e$$

Matrix Form of Above

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

FEM: Isoparametric Representation

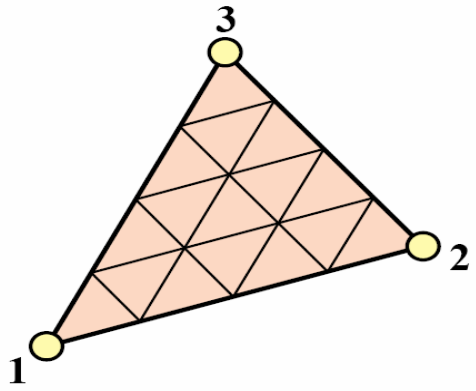
More Rows May be Added to Interpolate other Quantities from Node Values

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ h \\ T \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ u_{x1} & u_{x2} & \dots & u_{xn} \\ u_{y1} & u_{y2} & \dots & u_{yn} \\ h_1 & h_2 & \dots & h_n \\ T_1 & T_2 & \dots & T_n \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_n^e \end{bmatrix}$$

thickness h
temperature T

FEM: Isoparametric Representation

The Linear Triangle

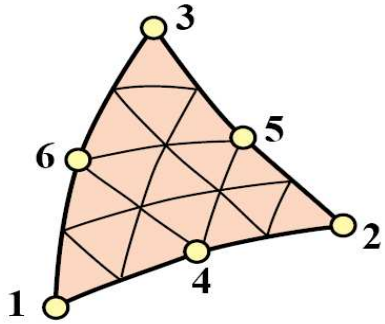


$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{x1} & u_{x2} & u_{x3} \\ u_{y1} & u_{y2} & u_{y3} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

$$N_1^e = \zeta_1, \qquad N_2^e = \zeta_2, \qquad N_3^e = \zeta_3$$

FEM: Isoparametric Representation

The Quadratic Triangle

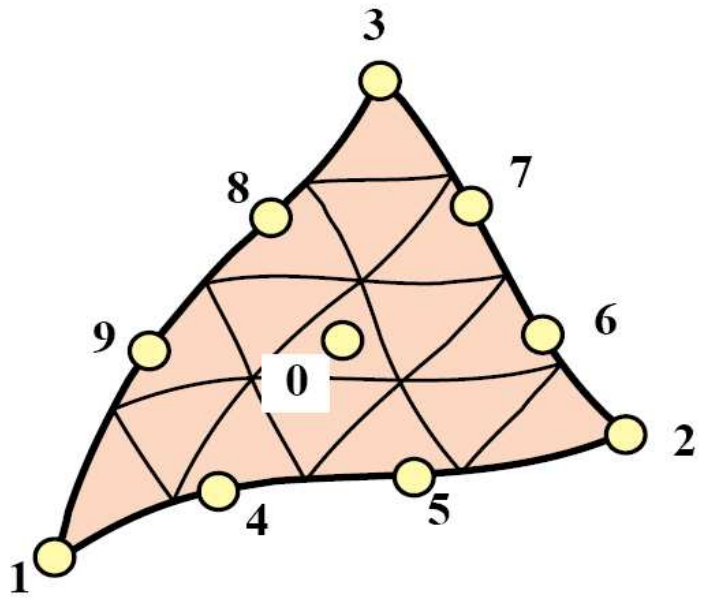


$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ N_5^e \\ N_6^e \end{bmatrix}$$

$$N_1^e = \zeta_1(2\zeta_1 - 1) \qquad N_4^e = 4\zeta_1\zeta_2$$
$$N_2^e = \zeta_2(2\zeta_2 - 1) \qquad N_5^e = 4\zeta_2\zeta_3$$
$$N_3^e = \zeta_3(2\zeta_3 - 1) \qquad N_6^e = 4\zeta_3\zeta_1$$

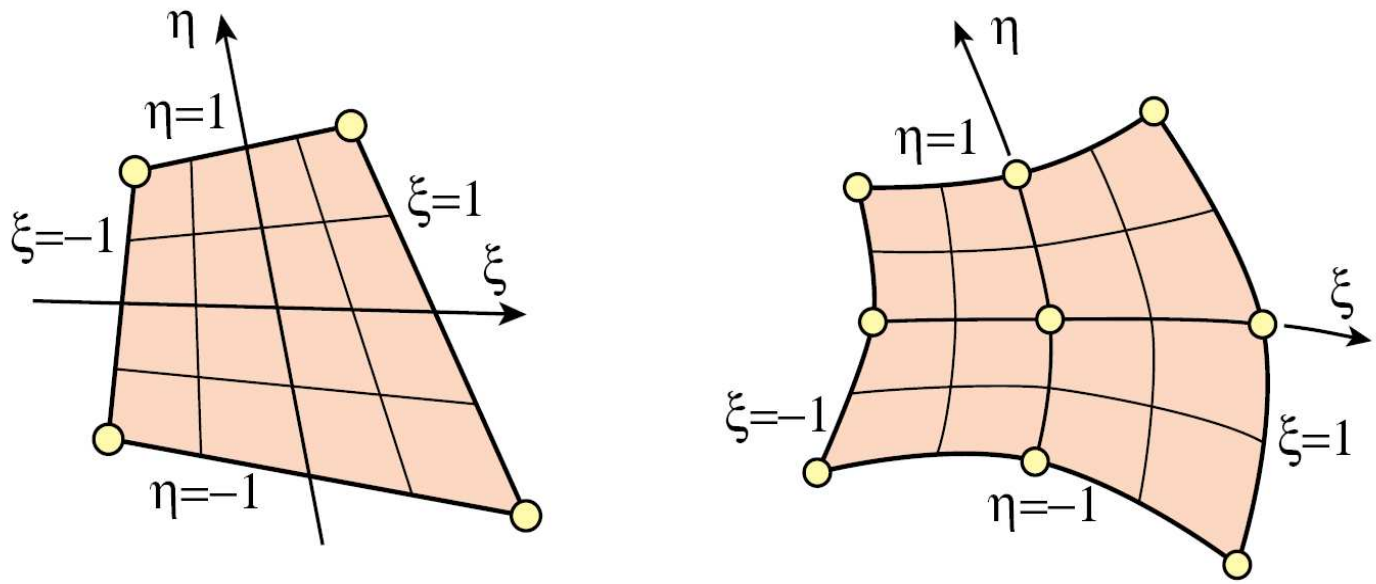
FEM: Isoparametric Representation

The Cubic Triangle



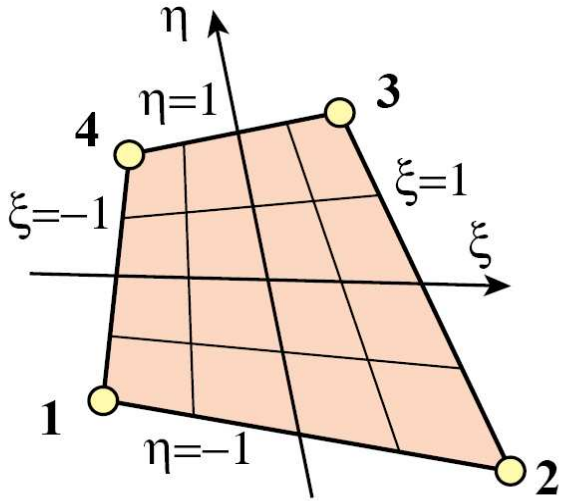
FEM: Isoparametric Representation

Quadrilateral Coordinates ξ, η



FEM: Isoparametric Representation

4-Node Bilinear Quadrilateral

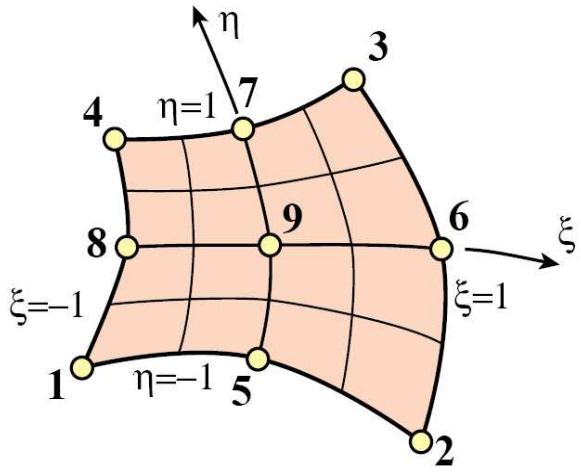


$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \end{bmatrix}$$

$$\begin{aligned} N_1^e &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2^e &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3^e &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4^e &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}$$

FEM: Isoparametric Representation

9 Node Biquadratic Quadrilateral



$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} & u_{x5} & u_{x6} & u_{x7} & u_{x8} & u_{x9} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} & u_{y5} & u_{y6} & u_{y7} & u_{y8} & u_{y9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix}$$

$$N_1^e = \tfrac{1}{4}(1-\xi)(1-\eta)\xi\eta \qquad N_5^e = -\tfrac{1}{2}(1-\xi^2)(1-\eta)\eta$$
$$N_2^e = -\tfrac{1}{4}(1+\xi)(1-\eta)\xi\eta \qquad N_6^e = \tfrac{1}{2}(1+\xi)(1-\eta^2)\xi \qquad N_9^e = (1-\xi^2)(1-\eta^2)$$
$$\cdots \qquad \cdots$$

FEM: Convergence Requirements

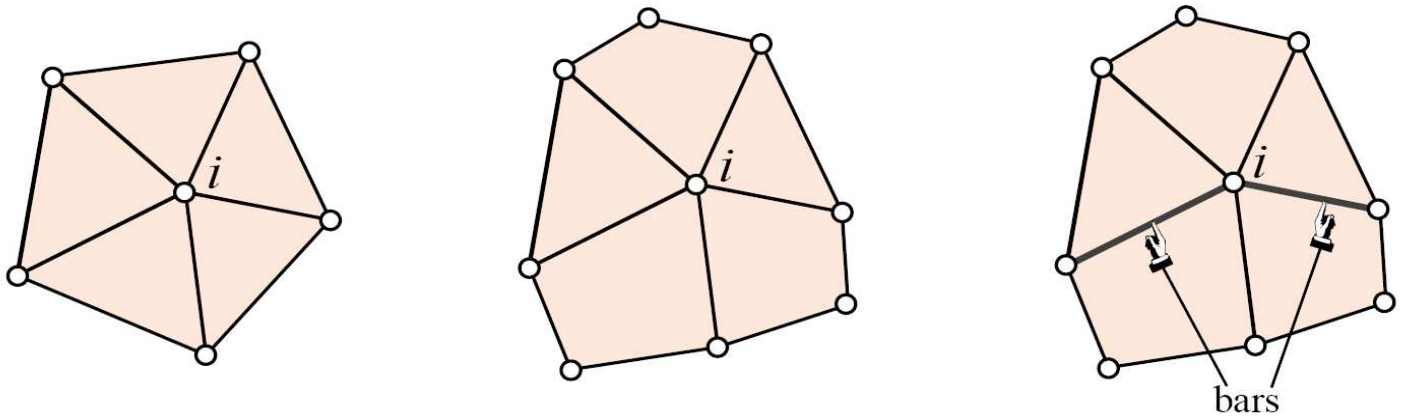
Convergence Requirements for Finite Element Discretization

Convergence: discrete (FEM) solution approaches the analytical (math model) solution in some sense

Convergence = Consistency + Stability

Element Patches

A ***patch*** is the set of all elements attached to a given node:



A finite element ***patch trial function*** is the union of shape functions activated by setting a degree of freedom at that node to unity, while all other freedoms are zero. A patch trial function "propagates" only over the patch, and is zero beyond it.

FEM: Convergence Requirements

Completeness & Compatibility in Terms of *m*

Completeness

The ***element shape functions*** must represent exactly all polynomial terms of order $\leq m$ in the Cartesian coordinates. A set of shape functions that satisfies this condition is call *m*-complete

Compatibility

The ***patch trial functions*** must be $C^{(m-1)}$ continuous between elements, and C^{m} piecewise differentiable inside each element

FEM: Convergence Requirements

Plane Stress: $m = 1$ in **Two Dimensions**

Completeness

The ***element shape functions*** must represent exactly all polynomial terms of order ≤ 1 in the Cartesian coordinates. That means any *linear polynomial* in x, y with a *constant* as special case

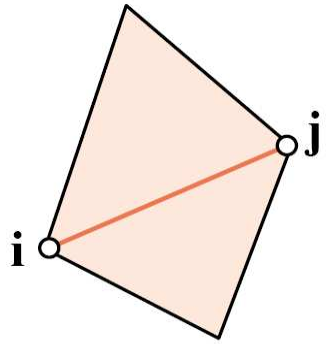
Compatibility

The ***patch trial functions*** must be C^0 continuous between elements, and C^1 piecewise differentiable inside each element

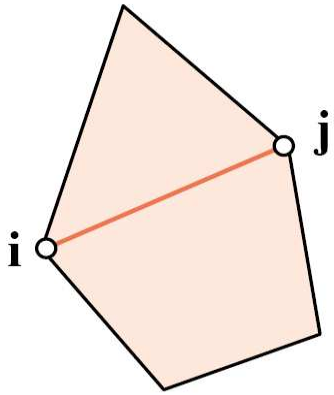
FEM: Convergence Requirements

Interelement Continuity is the Toughest to Meet

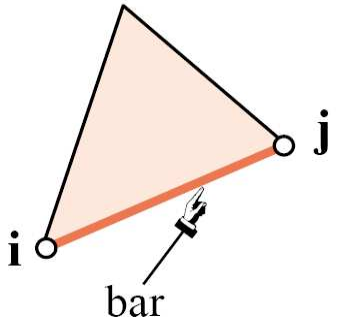
Simplification: for ***matching meshes*** (defined in Notes) it is enough to check compatibility between a ***pair of adjacent elements***:



Two 3-node linear triangles



One 3-node linear triangle and one 4-node bilinear quad

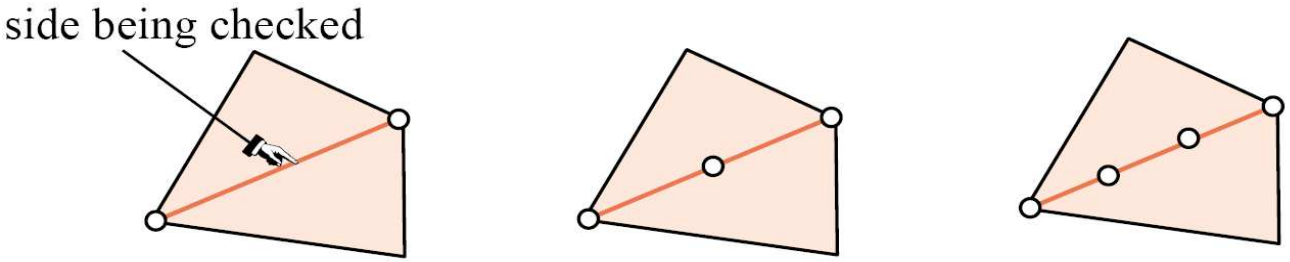


One 3-node linear triangle and one 2-node bar

FEM: Convergence Requirements

Side Continuity Check for Plane Stress Elements with Polynomial Shape Functions in Natural Coordinates

Let k be the number of nodes on a side:



$k = 2$ $k = 3$ $k = 4$

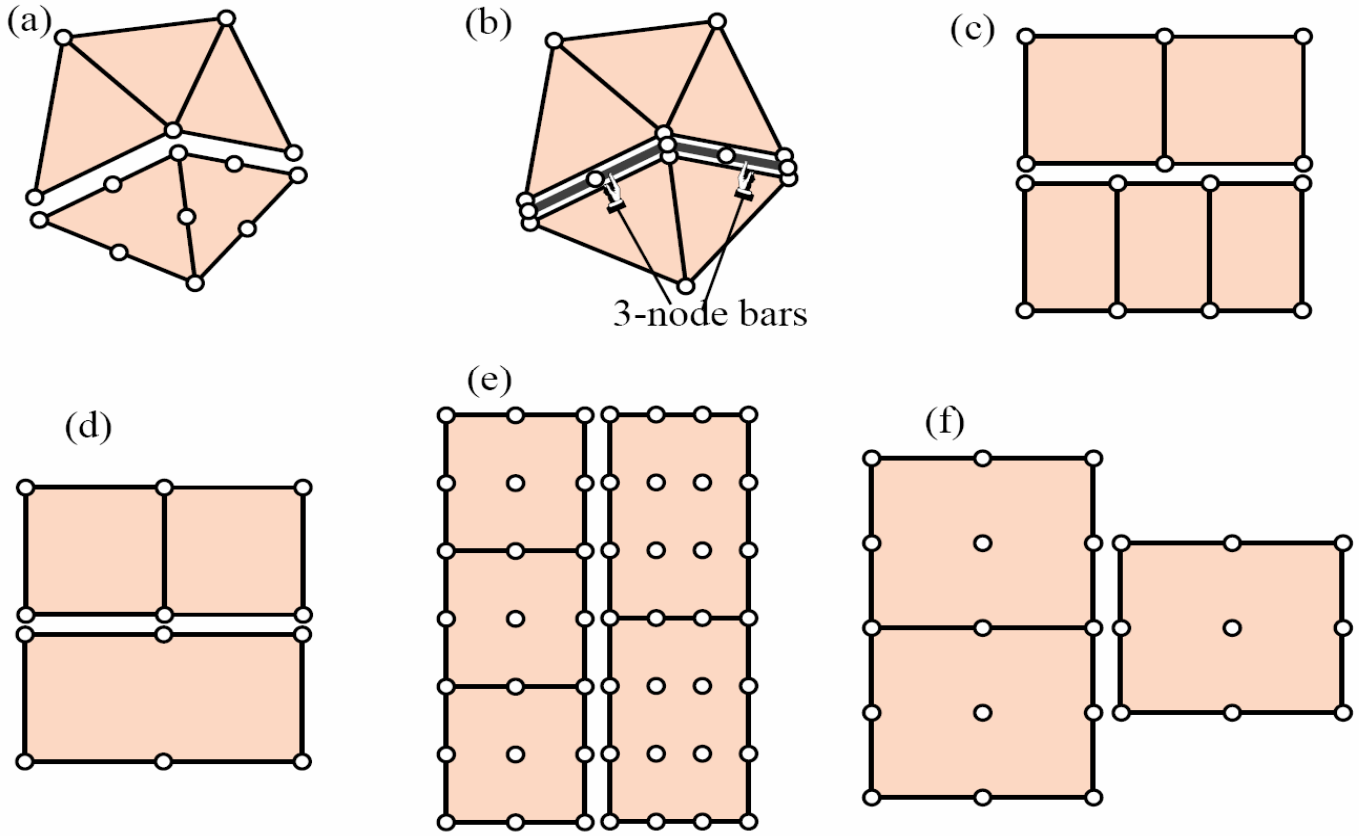
The variation of each element shape function along the side must be of polynomial order $k - 1$

If ***more, continuity is violated***

If ***less, nodal configuration is wrong*** (too many nodes)

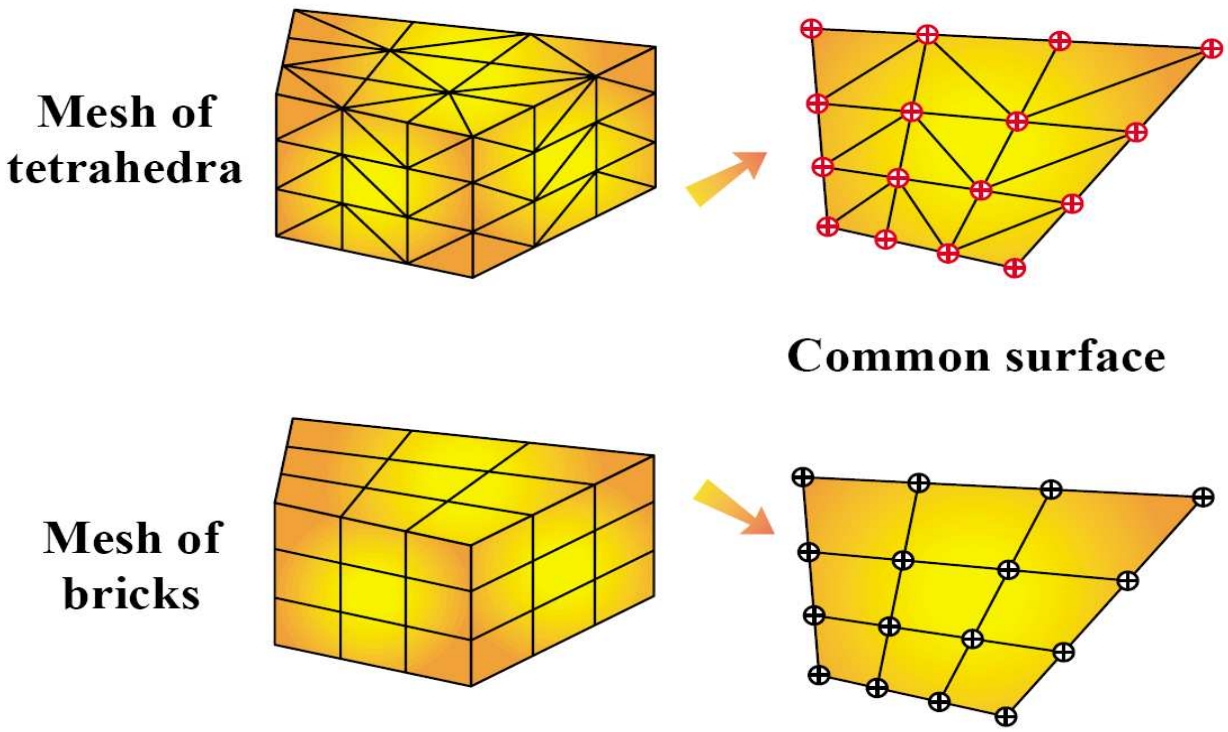
FEM: Convergence Requirements

2D Nonmatching Mesh Examples



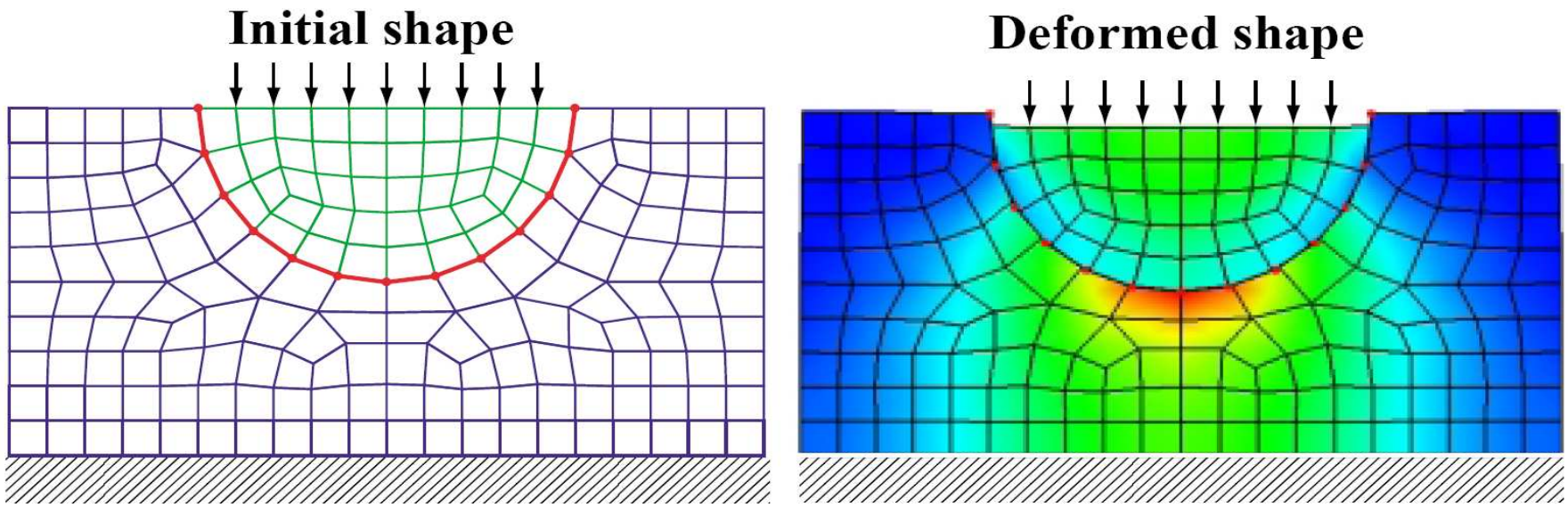
FEM: Convergence Requirements

3D Nonmatching Mesh Example



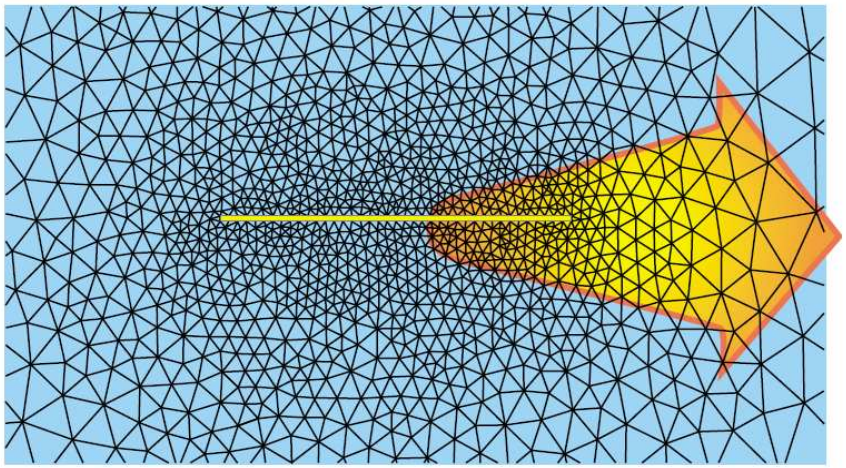
FEM: Convergence Requirements

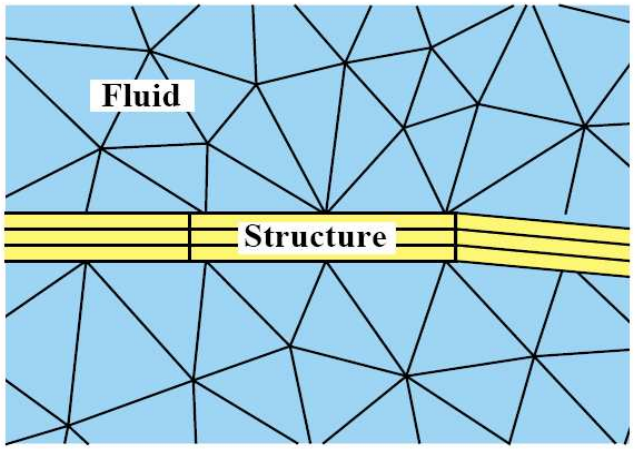
Nonmatching Meshes in Contact-With-Slip
(a Geometrically Nonlinear Problem)



FEM: Convergence Requirements

Nonmatching Meshes in FSI Problem





FEM: Convergence Requirements

Stability

Rank Sufficiency

The discrete model must possess the same solution uniqueness attributes of the mathematical model
For displacement finite elements:

the rigid body modes (RBMs) must be preserved
no zero-energy modes other than RBMs

Can be tested by looking at the **rank** of the stiffness matrix

Positive Jacobian Determinant

The determinant of the Jacobian matrix that relates Cartesian and natural coordinates must be everywhere ***positive*** within the element

Recommended Books for Linear FEM

Basic level (reference): Zienkiewicz & Taylor (1988), Vols I (1988), II (1993). A comprehensive upgrade of the 1977 edition. Primarily an encyclopedic reference work that provides a panoramic coverage of FEM, as well as a comprehensive list of references. Not a textbook. A fifth edition has appeared.

Basic level (textbook): Cook, Malkus & Plesha (1989); this third edition is fairly comprehensive in scope and up to date although the coverage is more superficial than Zienkiewicz & Taylor.

Intermediate level: Hughes (1987). It requires substantial mathematical expertise on the part of the reader Recently reprinted by Dover..

Mathematically oriented: Strang & Fix (1973). Most readable mathematical treatment although outdated in several subjects.

Most fun (if you like British "humor"): Irons & Ahmad (1980)

Best value for the $$$: Przemieniecki (Dover edition 1985, ~$16). Although outdated in many respects (e.g. the word "finite element" does not appear in this reprint of the original 1966 book), it is a valuable reference for programming simple elements.

Comprehensive web search engine for out-of print books: **http://www3.addall.com**